

A Kaluza-Klein Monopole^{*}

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Abstract

By adding the trivial term $-dt^2$ one can convert the four-dimensional (positive-definite) NUT line element into a static solution of the five-dimensional vacuum Einstein equations. Interpreted *à la* Kaluza-Klein, the solution describes a monopole with a charge-to-mass ratio of $\sqrt{2}$ in rationalized Planck units. Although it is a perfectly regular 5-geometry, it appears singular from a 4-dimensional perspective.

Present-day physics exhibits two independent kinds of symmetries: the geometrical symmetries expressing the interchangeability of the points of spacetime, and the internal symmetries expressing an interchangeability among certain of the material structures which can occur at a single spacetime point. The aim of so-called Kaluza-Klein theories is to unify these types of symmetry by hypothesizing that the apparently internal symmetries are in reality also geometrical. One posits the existence of extra (but very compact) dimensions of spacetime and interprets gauge transformations as generalized rotations taking place within these extra dimensions.

In the prototype Kaluza-Klein theory,¹ the fifth spacetime coordinate was largely unphysical, since the metric components were forbidden to depend on it. Even in the absence

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¹ Th. Kaluza, *Sitzungsber. Preuss. Akad. Wiss. Math. Phys. Kl.* 1921 pp. 966-972 (1921); O. Klein, *Z. Physik* **37**: 895-906 (1926).

of such a restriction, however, a cynic might still choose to dismiss the use of extra dimensions as a physically meaningless artifice for producing interesting 4-lagrangians. Short of direct exploration of the “internal” geometry, the best antidote to such cynicism might be the discovery of a particle whose hypothesized structure *does not admit* of interpretation in terms of 4-dimensional fields. In fact, such structures exist, and we will see below that one which occurs in the purely electromagnetic theory can be interpreted as a magnetic monopole carrying the Dirac charge.

The Lagrangian

For the abelian $[U(1)]$ Kaluza-Klein theory, spacetime is a 5-manifold 5M with metric g_{AB} of signature $- + + + +$. To descend to four dimensions one introduces a vector field K^A whose orbits are circles ($\exp 2\pi K = 1$) and assumes that, at least to a first approximation, K is a Killing vector for g_{AB} . The 5-metric then decomposes as

$$g_{AB} = \gamma_{AB} + A_A A_B , \quad (1)$$

where γ_{AB} is a degenerate metric of signature $0 - + + +$ such that $\gamma_{AB} K^B = 0$. The one-form A is well defined wherever $K \neq 0$ and related to K^A by $A_A = K_A/\lambda = g_{AB} K^B/\lambda$, where $\lambda = (g_{AB} K^A K^B)^{1/2}$ is the radius of the “internal circle”.

The effective 4-manifold 4M is now the *quotient* of 5M with respect to the action of K : Each circular orbit of K becomes a single point in 4M . Because of the assumption that K is a symmetry, nothing really depends on the fifth dimension; and g_{AB} is equivalent to a set of three distinct fields defined on 4M , a scalar, a tensor, and a vector, deriving respectively from λ , γ_{AB} , and A_A . At spatial infinity λ will approach a constant value λ_∞ , and we take the 4-scalar field to be

$$\omega = -\frac{1}{2} \ln(\lambda/\lambda_\infty) . \quad (2)$$

To define the 4-metric $\tilde{g}_{\mu\nu}$ (using Greek indices for 4M) notice that there is a unique metric $\gamma_{\mu\nu}$ whose pullback to 5M is γ_{AB} and put

$$\tilde{g}_{\mu\nu} = (\lambda/\lambda_\infty) \gamma_{\mu\nu} = e^{-2\omega} \gamma_{\mu\nu} . \quad (3)$$

In a coordinate system with x^5 parameterizing the internal dimension and with $K^A = (0, 0, 0, 0, 1)$ the components of g_{AB} are independent of x^5 and our definitions so far read

$\lambda = (g_{55})^{1/2}$, $\tilde{g}_{\mu\nu} = (\lambda/\lambda_\infty)(g_{\mu\nu} - g_{\mu 5}g_{\nu 5}/g_{55})$. Finally, we will need the vector potential $\tilde{A}_\mu \equiv A_\mu/\lambda$. Unlike ω and $\tilde{g}_{\mu\nu}$, \tilde{A}_μ is gauge dependent — it changes under coordinate transformations $x^5 \rightarrow x^5 + \theta(x^\mu)$ — but of course

$$\tilde{F}_{\mu\nu} \equiv (\text{curl } \tilde{A})_{\mu\nu} \equiv \partial_\mu \tilde{A}_\nu - \partial_\nu \tilde{A}_\mu \quad (4)$$

is not, and can be defined invariantly as the two-form on 4M whose pullback to 5M is $F \equiv \text{curl}(A/\lambda)$.

With these definitions² the 5-lagrangian $(2k)^{-1}({}^5R)(-{}^5g)^{1/2}$ leads to a 4-action

$$S = S_\infty + \frac{1}{\kappa} \int \left(\frac{1}{2} \tilde{R} - 3|\tilde{\nabla}\omega|^2 - \tilde{\nabla}^2\omega - \frac{1}{8} e^{-6\omega} \lambda_\infty^2 \tilde{F}_{\mu\nu} \tilde{F}^{\mu\nu} \right) d\tilde{V}, \quad (5)$$

where \tilde{R} is the scalar curvature of $\tilde{g}_{\mu\nu}$ (with respect to which all contractions are done), and $\kappa = k/2\pi\lambda_\infty$ plays the role of the four-dimensional gravitational constant $8\pi G$. The integral S_∞ at spatial infinity would be needed in deducing the total energy from first principles, but we may neglect it for what follows.

A monopole solution

Since the 5-lagrangian is just the scalar curvature, the five-dimensional field equations are just ${}^5R_{AB} = 0$. Consequently any signature-(++++) solution g_{AB} of 4-gravity can be extended to a static Kaluza-Klein solution g_{AB} simply by setting $g_{00} = -1$, $g_{a0} = 0$ (where $a, b = 1, 2, 3, 5$). For example the flat metric derived from $g_{ab} = \delta_{ab}$ by periodically identifying $x^5 = x^5 + 2\pi\lambda_\infty$ gives rise in this way to the Kaluza-Klein equivalent of Minkowski spacetime.

Applying this procedure to the so-called Taub-NUT instanton yields a 5-metric which can be written conveniently³ in terms of Cartesian coordinates t, x, y, z, w for \mathbb{R}^5 . In terms

² and with the following conventions: $c \equiv 1$, $[\nabla_M, \nabla_N]V^A = R^A{}_{BMN}V^B$, $R_{MN} = R^A{}_{MAN}$.

³ The Taub-NUT metric is described in A. Taub and C.W. Misner, *Zh. Eksp. Teor. Fiz.* **55**: 233 (1968) [*Sov. Phys. JETP* **28**: 122 (1969)]. The present metric corresponds to the choice $\ell = im$ (with ‘ ρ ’ denoting their coordinate t).

of $q^2 = x^2 + y^2 + z^2 + w^2$, $\rho = 2 \ln q$, and $\sigma_z = 2q^{-2}(zdw - wdz + xdy - ydx)$ (with cyclic permutations for σ_x and σ_y), the full Kaluza-Klein metric is

$$ds^2 = -dt^2 + U^{-1}d\rho^2 + 4m^2U\sigma_z^2 + (\rho^2 - m^2)(\sigma_x^2 + \sigma_y^2), \quad (6)$$

where $U = (\rho - m)/(\rho + m)$. In (6) there is an apparent singularity at $\rho = m$. However, a simple radial coordinate transformation and redefinition of ρ brings ds^2 to a form valid for all t, x, y, z, w , and in particular shows that (for any $m > 0$) the topology of the solution is simply \mathbb{R}^5 . The metric g_{AB} thereby assumes the form⁴

$$-\partial t^2 + \left(1 + \frac{m}{\rho}\right)^2 \partial r^2 + \left(\frac{2mr}{\rho + m}\right)^2 \sigma_z^2 + r^2(\sigma_x^2 + \sigma_y^2), \quad (7)$$

where $r^2 \equiv x^2 + y^2 + z^2 + w^2$ and

$$\rho^2 \equiv r^2 + m^2. \quad (8)$$

An equivalent expression in terms of Euler angles for the 3-sphere can be obtained by substituting³

$$\sigma_z = -\partial\psi - \cos\theta \partial\varphi, \quad (9)$$

$$\sigma_x^2 + \sigma_y^2 = \partial\theta^2 + (\sin\theta)^2\partial\varphi^2 \equiv \partial\Omega^2, \quad (10)$$

with $0 \leq \theta \leq \pi$ and with φ and ψ having periods 2π and 4π respectively⁵. That (7) is regular at $r = 0$ may not be immediately recognizable, but direct computation reveals that near $r = 0$ it reduces to $-\partial t^2 + 4(\partial w^2 + \partial x^2 + \partial y^2 + \partial z^2)$, whose smoothness is manifest.

⁴ Here ‘ ∂t ’ denotes the one-form ∂_{At} ; ‘ ∂t^2 ’ stands for $\partial t \otimes \partial t$, *i.e.* $\partial_{At} \partial_{Bt}$; and σ_x , σ_y , and σ_z are to be understood as one-forms defined in \mathbb{R}^4 .

⁵ To remove the familiar polar coordinate singularities one makes the unfamiliar identifications at $\theta = \pm\pi$, $(\theta, \varphi, \psi) = (\theta, \varphi + \Delta, \psi \mp \Delta)$, Δ arbitrary. *Added note:* Though not exactly false, the statement in the text that φ and ψ have different periods is misleading, since the situation is perfectly symmetric between the two coordinates, as far as 5M is concerned. It would be better to say that one quotients the φ - ψ -plane by the lattice in \mathbb{R}^2 generated by the pair of vectors $(2\pi, \pm 2\pi)$. One can then choose any convenient fundamental domain to represent the quotient, the choice $\varphi \in [0, 2\pi)$ and $\psi \in [0, 4\pi)$ being only one possibility among many. Notice in this connection that incrementing φ by 2π at *fixed* ψ returns one to the antipodal point on the fiber from which one set out. Notice also that with most choices of fundamental domain, φ is not confined to the range $[0, 2\pi]$ and must be reduced modulo 2π in order to become the usual azimuthal angle on 4M .

Equation (7) defines our five-dimensional soliton solution. It is manifestly static, already in the form (1) with $A = 2mr/(\rho + m)\sigma_z$, and in $t, r, \theta, \varphi, \psi$ coordinates clearly admits $K = 2 \partial/\partial\psi$ as Killing vector with period 2π (*i.e.* $\psi/2$ can be used to parameterize the internal circles⁶). It follows that

$$\lambda = \frac{4mr}{\rho + m} . \quad (11)$$

For $r \gg m$, the metric approaches $-\partial t^2 + \partial r^2 + r^2 \partial \Omega^2 + 4m^2 \sigma_z^2$, which locally looks like a ‘‘Kaluza-Klein Minkowski space’’ with internal radius $8\pi m$; in particular $\lambda_\infty = 4m$.

Proceeding from these formulas it is easy to pass to the 4-description of our metric. The quotient manifold 4M is naturally coordinatized by t, r, θ, ϕ and is thus topologically \mathbb{R}^4 . The true spacetime 5M is everywhere a fibre bundle over 4M *except at the origin* $r = 0$, where $\lambda = 0$ and the $U(1)$ fibre shrinks to a point. From (10) $\gamma_{\mu\nu}$ is $-\partial t^2 + (1 + m/\rho)^2 \partial r^2 + r^2 \partial \Omega^2$, whence the quotient metric $\tilde{g}_{\mu\nu}$ of (3) is (with the help of (8))

$$-\frac{\rho - m}{r} \partial t^2 + \frac{r}{\rho} \left(1 + \frac{m}{\rho}\right) \partial r^2 + (\rho - m) r \partial \Omega^2 . \quad (12)$$

From (2) and (11) we see that

$$\omega = \frac{1}{2} \ln \frac{\rho + m}{r} . \quad (13)$$

Finally $\text{curl}(A/\lambda) = \frac{1}{2} \text{curl} \sigma_z = \frac{1}{2} \sin \theta \partial \theta \wedge \partial \varphi$, which translates immediately to 4M to give for the electromagnetic field tensor

$$\tilde{F} = \frac{1}{2} \sin \theta \partial \theta \wedge \partial \varphi . \quad (14)$$

Observe that $\tilde{g}_{\mu\nu}$, ω and \tilde{F} all become singular at $r = 0$, as could have been anticipated from the breakdown there of the local product structure of 5M .

Mass and charges of the solution

To justify calling this solution a monopole let us compute its electromagnetic charges. The total magnetic flux through any sphere about the origin is, from (14),

$$\oint \tilde{B} \cdot dS = \oint \frac{1}{2} \tilde{F}_{\mu\nu} d\Sigma^{\mu\nu} = \oint \frac{1}{2} \sin \theta d\theta d\phi = 2\pi ,$$

⁶ Except near $\theta = 0, \pi$, where $\psi \pm \varphi$ can be used. [Small addition here from published version, which omitted $\theta = 0$.]

as could have been inferred directly from the fact that the NUT metric has one twist⁷ when regarded as a $U(1)$ bundle over 4M . However, this flux is not immediately the magnetic charge because the coefficient of $F^{\mu\nu}F_{\mu\nu}$ in (5) differs from the customary $-\frac{1}{4}$, the correctly normalized magnetic charge⁸ being rather $Q_B = 2\pi\lambda_\infty(2\kappa)^{-1/2}$. In units such that $k = 2\pi\lambda_\infty = 1$ (then $\hbar \neq 1!$), which we can designate “KK units”, this becomes $Q_B = 1/\sqrt{2}$.

As for the total *electric* flux, $\oint \tilde{E} \cdot dS$, it vanishes because the absence of ∂t in (14) means that E itself vanishes. Notice incidentally that the Dirac condition $eQ_B/h = \text{integer}$ (where $h = 2\pi\hbar$) is verified with $n = 1$ since the unit of electric charge is $e = (2\kappa)^{1/2}\hbar/\lambda_\infty$.

The monopole’s scalar charge is somewhat easier to compute. According to (13) and (8), $\omega \approx m/2r$ at large radii and is therefore the field of a source of strength $2\pi m = (\pi/2)\lambda_\infty$. Again, however, ω should be rescaled by $(6/\kappa)^{1/2}$ to bring its contribution to (5) to the usual form. The resulting scalar charge is $Q_S = (3/2\kappa)^{1/2}\pi\lambda_\infty$ or $\sqrt{3/8}$ in KK units.

Finally, what about the mass? With the definitions of this paper we can deduce the total energy M unambiguously⁹ from the behavior of \tilde{g}_{00} as $r \rightarrow \infty$. From (12), $-\tilde{g}_{00} \sim 1 - m/r$, whence $M = 4\pi m/\lambda_\infty = \pi\lambda_\infty/\kappa$ or $\frac{1}{2}$ in KK units. To summarize, then, the mass and charges in KK units are¹⁰

$$M = \frac{1}{2}, \quad Q_B = \sqrt{\frac{1}{2}}, \quad Q_S = \sqrt{\frac{3}{8}}.$$

⁷ S. Ramaswamy and A. Sen, *J. Math. Phys.* **22**: 2621 (1981)

⁸ In so called Heaviside units for which the force between two charges is $Q_1Q_2/4\pi r^2$.

⁹ There can be only one conserved quantity whose “matter” contributions (from the stress-energy tensors for ω and $F_{\mu\nu}$) are correctly normalized in regions of small gravitational field. It was for this convenience in identifying M that we chose to use (3) and not just γ as the 4-metric.

¹⁰ Thus the present solution has an electromagnetic charge greater than that of any Reissner-Nordstrom metric of the same mass.

Incidentally the value of Q_B implies that bound states uniting the monopole with charge e excitations of g_{AB} would be fermions.¹¹

Discussion

In no four-dimensional theory with a scaling symmetry can there be a stationary solution of nonzero energy.¹² How then do the Kaluza-Klein equations admit a solution (7) which is not only stationary, but everywhere regular and without event horizon? Two features seem essential. The inescapable five-dimensionality of the metric prevents any 4-theorems from being applied directly; and the fixed value of the internal radius λ_∞ in effect introduces a fundamental length which then prevents the 5-dimensional analog theorem from applying. This seems quite similar to the way in which a boundary condition on the Higgs field allows monopole solutions to persist in the Prasad-Sommerfield limit, even though the scale-invariance-breaking potential itself goes to zero. Here, however, there is no Higgs field and no symmetry-breaking.

In the same way as we used the NUT metric, each of the known “ALF instantons” could be used to construct a static Kaluza-Klein solution. For example, the multi-Taub-NUT solutions become multi-monopole solutions and the Schwarzschild instanton of “mass” M yields a neutral solution of total energy $M/2$ which seems to describe a delicately balanced “bubble” of the type discussed in Ref. thirteen.¹³ By a slight generalization of the present procedures (to allow g_{0A} to vary) the Israel-Wilson solutions may also turn into 5-dimensional vacuum solutions.

¹¹ See A.O. Barut, *Phys. Rev. D* **10**: 2712 (1974); R.D. Sorkin, *Phys. Rev. D* **27**: 1787 (1983).

¹² This follows from a theorem in B.F. Schutz and R.D. Sorkin, *Ann. Phys. (N.Y.)* **107**: 1 (1977), assuming that the theory is defined by an “unconstrained” action principle.

¹³ E. Witten, *Nucl. Phys. B* **195**: 481 (1982). Notice though that with the metric $\tilde{g}_{\mu\nu}$ the bubble becomes a point singularity and there is actually no “hole” in the apparent spacetime.

Of course, all these solutions are physically unrealistic insofar as $U(1)$ is not the gauge group realized in nature.¹⁴ But in order that there be analogs it may suffice that the “correct” G contain $U(1)$ as a subgroup, which is virtually certain.

In a way it is surprising that so many seemingly distinct particle-like structures can exist in what seem to be fundamental theories.¹⁵ One might wish that at least some of these possibilities could be, if not excluded, then at least reduced to some of the others. In the context of the present theory, for example, there is no reason why the charge carried by the monopole should be labeled “magnetic”. It might equally well be deemed “electric” and \tilde{A}_μ treated as a potential for the dual field $*F_{\mu\nu}$. With this re-identification — or even without it — it becomes important to understand how the effective dual potential needed to express the mutual interaction of such monopoles can arise. Once this is done,¹⁶ it may appear that some or all of what today seem to be elementary structures (including spacetime itself) arise in a similar way from “sub-Planck-scale” quantum processes.

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¹⁴ They are probably also quantum mechanically unstable via the mechanism of Ref. 13. However, one should at least decide which ones are classically stable, say with the methods of G.W. Gibbons and C.M. Hull, *Phys. Lett.* **109B**: 190 (1982).

¹⁵ For some examples see E. Lubkin, *Annals of Physics (N.Y.)* **23**: 233-283 (1963); R. Sorkin, *J. Phys. A* **10**: 717-725 (1977) and **11**: 795 (1978); D. Finkelstein, *J. Math. Phys.* **7**: 1218 (1966).

¹⁶ B. Hu, in *Proceedings of the Guangzhou Conference on Particle Physics* (Science Press, Beijing, 1980).