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Quantum Dynamics without the Wave Function*

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Abstract

When suitably generalized and interpreted, the path-integral offers an alternative to the more familiar quantal formalism based on state-vectors, selfadjoint operators, and external observers. Mathematically one generalizes the path-integral-as-propagator to a *quantal measure* μ on the space Ω of all "conceivable worlds", and this generalized measure expresses the dynamics or law of motion of the theory, much as Wiener measure expresses the dynamics of Brownian motion. Within such "histories-based" schemes new, and more "realistic" possibilities open up for resolving the philosophical problems of the state-vector formalism. In particular, one can dispense with the need for external agents by locating the predictive content of μ in its sets of measure zero: such sets are to be "precluded". But unrestricted application of this rule engenders contradictions. One possible response would remove the contradictions by circumscribing the application of the preclusion concept. Another response, more in the tradition of "quantum logic", would accommodate the contradictions by dualizing Ω to a space of "co-events" and effectively identifying reality with an element of this dual space.

Reading the literature on "quantum foundations", you could easily get the impression that the problems begin and end with non-relativistic quantum mechanics. Perhaps most authors have limited themselves to this special case because they thought that no essentially new philosophical questions arose in relativistic quantum field theory and quantum gravity,

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or perhaps they merely felt that non-relativistic quantum mechanics was hard enough to interpret without bringing in the further complications of relative simultaneity and a dynamical causal structure. This feeling is true, no doubt, but it could also be that an over-emphasis on the nonrelativistic case has led people in directions they might not have taken had they thought in a broader context.

Thus, for example, the concept of "subsystem" takes center stage in many interpretations of the quantum formalism. In "chemistry" (the theory of nuclei, electrons, and Coulombic interaction), one could perhaps construe a subsystem as a definite collection of particles, but what could it mean in the context of quantum field theory in curved spacetime, say? Another example is the concept of state-vector or "wave function", which seems to be tied (in spirit if not always in the letter) to a particular moment of time, while its unitary "evolution" takes place between designated spacelike hypersurfaces. Locating one's basic mathematical structure on a hypersurface is already uncomfortable in a relativistic spacetime, but in relation to quantum gravity, it raises not only technical* but severe conceptual difficulties (the notorious "problems of time").

For reasons such as these, anyone who thinks seriously about quantum gravity, must at some point ask themself whether the notion of "history" doesn't furnish a better starting point than the Hilbert spaces, state-vectors and operator-observables that are so familiar from textbooks. Or to pose the question another way: Is quantum mechanics about the wave function and its unitary evolution or is it about things like electrons, quarks and electromagnetic fields? And if the latter, then is an electron to be described three-dimensionally as (for example) a time-dependent point of space, or four-dimensionally as a world line (not necessarily completed)? A "realistic" and "spacetime" approach of the sort suggested by the last two sentences may be termed "histories-based". Such an approach will of course diverge technically from the state-vector approach, but more importantly

* The technical difficulties are especially severe in the context of causal set theory, where the obvious analog of a spacelike hypersurface is a maximal antichain. But since an antichain has by definition no intrinsic structure of its own, one lacks an analog of the induced metric, which in the continuum is supposed to serve as the argument of the quantum gravitational wave function. Extrinsically induced structures do exist and can be very useful for kinematic purposes as in [1]; but they seem too ad hoc to enter into a basic dynamical law.

for this paper, I'm claiming that it also differs at the purely philosophical, interpretive level, and specifically that it suggests new solutions to some of the recurring philosophical puzzles that have sprung from the quantum formalism.

To produce a histories-based formulation of quantum theory means in essence to emancipate the path-integral from its role as an auxiliary mathematical device (a mere tool for computing propagators or S-matrix elements) and to make of it the fundamental dynamical object of the theory. To that end, one must bring out more explicitly what it is that the path integral really computes, and having done so, one must then complete the story by explaining how this newly conceived path integral can play the role played in classical mechanics by the "laws of motion" — how in particular it can make possible the manifold (though still incomplete) predictions that lend quantum mechanics its great practical utility.

Fortunately, much of this work of reformulation has already been carried out; and we can claim to know — or at least to have a very good candidate for — what this reconceived path-integral is in mathematical terms. It is a mathematical object that can be represented either as a sort of quadratic measure on Ω or as a Hermitian measure on $\Omega \times \Omega$, where Ω is the "sample space" in the sense of probability theory.

II. The path integral as a quadratic measure

Any theory for which a path integral can be defined possesses a *sample space* Ω over which the integration takes place;[†] but the kind of element γ that comprises Ω varies from theory to theory. In n-particle quantum mechanics, for example, each γ would be a set of n trajectories; while for a scalar field theory, it might be a real or complex function on spacetime. Often the members of Ω are called "fine-grained histories" or simply "histories"; sometimes they are called "paths" or "trajectories". From the perspective of classical logic, Ω is the space of "possible worlds", since each of its members represents as complete a description of physical reality as is conceivable in the theory.

[†] Fermionic path "integrals" are excepted here. As normally defined, they are not integrals at all. See however [2]

Now the path integral is normally conceived of as a complex measure on Ω (or more accurately, on subsets of Ω consisting of trajectories with fixed endpoints). As such, it serves to evolve wave functions by giving the net amplitude for (say) a particle to "propagate" from one spacetime point to another. But in quantum mechanics as ordinarily understood, such amplitudes, and indeed the wave function itself, are only intermediaries in the computation of probabilities. The object that yields these probabilities directly is what I will call the quantal measure μ, a function that assigns to a subset $A \subseteq \Omega$ a real number $\mu(A) \geq 0$. Since this characterization is a bit vague, let us examine it more closely in the context of nonrelativistic point-particle mechanics, where the expression for μ is relatively simple.

To that end, let A be a subset of Ω defined by properties of γ that refer only to its restriction to the time-interval $t \in [t_0, T]$; and let $\rho(q, \overline{q})$ be the density-matrix at time t_0. Then the quantal measure of A is given by [♭]

$$\mu(A) = \int\limits_{\gamma \in A^T} d\nu(\gamma) \int\limits_{\overline{\gamma} \in A^T} d\nu(\overline{\gamma})\ e^{iS(\gamma) - iS(\overline{\gamma})}\ \delta(\gamma(T), \overline{\gamma}(T))\ \rho(\gamma(t_0), \overline{\gamma}(t_0)) \ , \tag{1}$$

where A^T denotes the set of all truncated trajectories that can be derived from elements of A by restriction to $[t_0, T]$, $S(\gamma)$ is the action $\int L dt$ of γ, ν is some "base measure" on the space Ω^T of truncated histories with respect to which the integration is performed, and $\gamma(t)$ denotes the location (in configuration space Q) of γ at time t. (Notice that the "truncation time" T can be chosen arbitrarily, as long as it is late enough for it to be decided by then whether or not $\gamma \in A$.) It is then easy to check that, according to the standard quantum rules, if the result of a measurement is expressed by the position of a "pointer" at a given time, and if A_i is the set of histories for which the atoms composing the pointer are in the ith position at that time, then $\mu(A_i)$ is the probability of that experimental outcome. (Strictly speaking, this interpretation of $\mu(A_i)$ as an objective probability goes beyond the standard rules, which only tell us that $\mu(A_i)$ is the probability that the pointer would be found in the ith position if measured by an observer external to the entire experimental setup. However, our only purpose here is to gain some familiarity with μ and some intuition

[♭] When non-bosonic identical particles are involved, one must insert into (1) an additional global phase-factor $\chi(\gamma, \overline{\gamma})$ that depends on the topological class to which the combined path $\gamma \cup \overline{\gamma}$ belongs.

for its practical meaning. Later we will attempt a more general and direct interpretation of μ not based on the — essentially classical — concept of probability.)

Now the whole point of introducing the quantal measure was that it is histories-based and it stands on its own two feet, without needing to refer to the apparatus of operator algebras, expectation-values, state vectors, etc. that constitutes the terrain on which most discussions of quantum philosophy take place. We'd thus like to characterize μ in a general manner, without referring to any specific expression like (1) above. From a mathematical point of view, what makes quantum physics different from classical physics (in which I'm including stochastic processes) is its "quadratic nature": probabilities are squares of sums of elementary amplitudes, rather than simple sums of elementary probabilities. Physically, this corresponds to the phenomenon of *interference*, and the quadratic character simply expresses the fact that quantum interference involves pairs of alternatives, but never triples (except indirectly as induced by the pairwise interference). For the purposes of this article, the quadratic character is not absolutely essential, but as it expresses a very deep feature of quantum mechanics, is seems appropriate to dwell on it for a bit. This will also let me define a certain condition of "strong positivity" that seems to play an important role in the detailed working out of any interpretation of the quantal measure.

The quadratic nature of μ can be captured in two ways, either directly or in terms of a function D which is effectively a complex measure on $\Omega \times \Omega$. The former characterization [3] rests on the following sum rule for mutually disjoint subsets A, B, C of Ω, which generalizes the simple additivity of classical probabilities:

$$\mu(A \sqcup B \sqcup C) - \mu(A \sqcup B) - \mu(B \sqcup C) - \mu(A \sqcup C) + \mu(A) + \mu(B) + \mu(C) = 0 \ . \qquad (2)$$

(The symbol $\sqcup$ denotes disjoint union.) This sum-rule sits at level 2 of a hierarchy of sum-rules [3] limiting the "degree" of μ. A classical probability-measure μ is additive and resides at level 1; a quantal μ resides at level 2, which includes level 1 as a special case; etc. *

* A function μ satisfying (2) is quadratic in the following sense. One can view μ as a kind of "integral" taking 0-1-valued step functions to complex numbers; and then (2) is the condition that this integral extend to a homogeneous quadratic form on linear combinations of the step functions. (This does not contradict the fact that level 2 includes level 1. If μ is additive it will of course also extend to a homogeneous *linear* form on linear combinations of step functions.)

It bears emphasis that (2) is an unconditional relationship that can only fail if quantum mechanics in the most general sense fails.[†] It can be thought of concretely in terms of a three-slit diffraction experiment, and conversely one could perform such an experiment as a very direct null test of the validity of the basic quantal assumptions. More generally, any experiment in which three mutually exclusive alternatives were made to interfere would provide a similar null test.

The above sum-rule expresses the quadratic nature of μ very directly, but it is not as simple as one might wish. Fortunately, one can solve it identically with the ansatz,

$$\mu(A) = D(A, A) \tag{3}$$

where D (called for historical reasons a *decoherence functional*) will be required to fulfill the following axioms:

Hermiticity $D(A, B) = D(B, A)^*$
Additivity $D(A \sqcup B, C) = D(A, C) + D(B, C)$
Strong Positivity for any finite collection of subsets $A_1, A_2, \ldots A_n$, the $n \times n$ Hermitian matrix $M_{ij} = D(A_i, A_j)$ is positive semidefinite (it has no negative expectation values).

To illustrate these axioms, let me just mention a couple of lemmas they imply, which are useful in connection with the notion of preclusion that we will turn to shortly:

(i) $\mu(A) = 0 \Rightarrow D(A, B) = 0$, whatever B may be;

(ii) $\mu(A) = \mu(B) = 0 \Rightarrow \mu(A \cup B) = \mu(A \cap B)$.

(The first is a consequence of strong positivity; the second follows from the first together with a certain identity that generalizes the classical principle of "inclusion-exclusion".)

[†] Notice in this connection that whereas (1) produces complex amplitudes and is consistent only in the presence of unitarity, (2) presupposes neither of these features. (Conversely, it sheds no light on *why* these features occur in nature.)

Having made the acquaintance of the quantal measure and its associated decoherence functional, we now have the general form of a histories-based quantum theory. The kinematics of such a theory [♭] is expressed through its sample space Ω; its dynamics is expressed through μ and D. What is still lacking, however, is a finished conceptual bridge to take us from this formal structure to its physical meaning — in particular to clarify how one can base predictions on μ without falling back on the problematic notions of macroscopic system or external observer.

III. The preclusion concept: toward an interpretation of μ

We have already introduced the sample space Ω and the quantal measure μ, which assigns non-negative real numbers to certain [⋆] subsets of Ω. In the parlance of probability theory, such a subset is called an *event*, and I will follow this usage, although the phrase "potential event" might be less subject to misinterpretation. As subsets, the events combine with each other in such a way as to form a *Boolean algebra* $\mathfrak{A}$, and this algebraic structure will play a major role for us later in connection with "quantum logic".

We have already seen that for some — more or less well-defined — set of "instrument events" A, $\mu(A)$ can be interpreted as an experimental probability. This furnishes an important "boundary condition" that must ultimately be satisfied by any interpretation that we come up with, but could it be more than that? Suppose that one could identify within $\mathfrak{A}$ a subalgebra $\mathfrak{A}^{macro}$ consisting of "the macroscopic events", and that the restriction of μ to $\mathfrak{A}^{macro}$ could be proven to obey not only (2) but also the level-1 sum rule, $\mu(A \sqcup B) - \mu(A) - \mu(B) = 0$. In that case, μ would define a probability functional on $\mathfrak{A}^{macro}$ and one could try to maintain that this exhausted its physical meaning. In doing so, one would be adopting the standpoint of "decoherent" or "consistent" histories [4]. But in this direction one encounters a series of problems, of which the first and most troublesome

[♭] I almost wrote "the ontology of such a theory", but we will see that this would have been too hasty.

[⋆] Exactly which subsets of Ω belong to the domain of μ is a technical question whose answer will vary from theory to theory. In the special case where Ω is a finite set, all of its subsets would normally be included in the domain. (Classically, the subsets in $\mathrm{dom}(\mu)$ are said to be "measurable".)

at a practical level is the difficulty in coming up with a sufficiently precise and tractable definition of "macroscopic" (that is of the subalgebra $\mathfrak{A}^{macro}$). Lacking this, one risks obtaining either no interpretation at all (if no satisfactory subalgebra can be shown to exist) or a multitude of conflicting interpretations (if more than one $\mathfrak{A}^{macro}$ offers itself). One might also worry (with a touch of whimsy) that an interpretation of this sort denies a basic aspect of quantum mechanics because it postulates, essentially as a matter of definition, that alternative macroscopic events cannot interfere, even in principle. More serious to my mind, though, is the fact that, in itself, such an interpretation fails to offer us a picture of quantum reality (this being arguably the most important task of any interpretation). Or to the extent that it does suggest such a picture, that picture tends to deny the existence of the microworld. It leads us to identify reality not with an individual history γ, but with an element of $\mathfrak{A}^{macro}$, and this would withhold meaning from any statement referring to individual atoms or other forms of microscopic matter.

For reasons such as these, I do not think that we could rest content with any interpretation founded on a distinction between $\mathfrak{A}$ and $\mathfrak{A}^{macro}$, even if it could overcome the technical problems alluded to above. But is there an alternative? Can we draw predictive inferences from μ without invoking the word "macroscopic"? To do so, we need to move away from the notion of probability, and one way of doing so is to replace it by the notion of "precluded event", this being an event that "to all intents and purposes cannot happen". Indeed, probability is reducible to this notion on one view of the matter, whence to take preclusion as the basis of our interpretation of the quantal measure would only be to carry over to the quantal situation, a time-honored idea for interpreting μ in the classical world. The basic idea, then, is to reduce all predictions to statements of the form: "The event $A \in \mathfrak{A}$ has zero measure $\mu(A)$, therefore A is precluded". Such a statement applies as well to microscopic events as to macroscopic ones. The prediction which it makes is of a type that one might call "definite but incomplete".

IV. The antinomy and three responses to it

The problem with preclusion

The principle we are aiming to hold onto affirms that events of measure zero (or of sufficiently small measure) "do not happen". Implicit in any principle like this is some conception of reality, with respect to which one could say, in any given case, whether or not a given $A \in \mathfrak{A}$ did or did not actually happen; and the most obvious and straightforward conception is the following. Reality (that which exists or happens) would be a definite member γ_{true} of Ω, and the statement that an event $A \subseteq \Omega$ "did not happen" would simply mean that γ_{true} was not to be found among the elements of A. (If the event "rain today" is not happening, that simply means that the "actual world" γ_{true} is not among those "potential worlds" γ in which it is raining today.) This particular "logic of being" is so ingrained in us that it is at first hard to think of an alternative;[†] nevertheless it may well turn out that we need an alternative if we are to develop the preclusion concept in a satisfactory manner. The problem, in a nut shell, is that there are "too many preclusions".

An elementary illustration of the difficulty is furnished by the three-slit experiment referred to earlier, which we may idealize for present purposes as a source emitting spinless particles which impinge on a diffraction grating with three slits, labeled a, b and c. Let A [respectively B, C] be the set of γ such that the particle traverses slit a [respectively b, c] and arrives at p, where p is a spacetime region—idealized as a point—which is aligned with the central slit and which consequently sits within a "bright band" of the diffraction pattern. For such a p, the measure $\mu(A \cup B \cup C)$ of the set of all world lines that arrive at p is nonzero, corresponding to the fact that if we look for the particle at p, we will frequently find it there. On the other hand, we can choose the separation between the slits so that, when taken in pairs (a, b) or (b, c), the amplitudes cancel, and correspondingly, the measures $\mu(A \cup B)$ and $\mu(B \cup C)$ will also vanish. An unrestricted preclusion rule would then entail that the actual trajectory γ_{true} could belong neither to $A \cup B$ nor to $B \cup C$, whence it could not belong to $(A \cup B) \cup (B \cup C) = A \cup B \cup C$, whence it could not arrive at p at all—a false prediction. Although in this case, the particle could still go elsewhere, the Kochen-Specker setup gives rise to an example [5] where every possibility without exception would be ruled out by an unrestricted preclusion rule. A theory cannot be more self-contradictory than that!

[†] Fay Dowker has named as "Axiom 0" the assumption that one and only one member of Ω is realized.

First response: it's here, it's queer, get used to it!

In a certain sense the same paradox is already present in classical probability theory, where for example, each single Brownian path γ taken individually is of measure zero, and therefore "cannot occur" — or "almost surely" cannot occur , in the stock phrase that mathematicians favor. In strict logic, this would seem to be a blatant inconsistency since one γ really does occur, but it hasn't stopped people from applying the probability calculus to good effect. They have just grown used to such paradoxes and learned to live with them.

Could something similar happen with the paradoxes flowing from quantal preclusion? Granted, they are made quantitatively worse by the effects of interference (one needed an infinite number of preclusions to arrive at a contradiction in the Brownian motion example, but only two in the three-slit example), but could we nonetheless grow used to them and gradually cease to be troubled by them?[♭] If that were possible, we could embrace the unrestricted preclusion rule with no worse a conscience than we feel in connection with the paradoxes of ordinary probability theory. Perhaps future generations will think this way, but I personally would not bet on it. It might even turn out that, on the contrary, the effort to understand the quantal measure will end up clarifying the meaning of classical probability, given that the latter is but the classical shadow of the former.

Second response: limited preclusion

A second way out of the impasse might be to restrict the application of the preclusion rule sufficiently so that no contradictions remained among the surviving preclusions. In another place [6], I've written about this approach and its relationship to that of the decoherent historians. I will not attempt to reproduce those reflexions here, but only to append a few remarks on the problems and remaining possibilities of the approach, which takes inspiration from the apparent absence of contradictions among the preclusions induced by measurements. One seeks to abstract from measurement-situations some feature that

[♭] Bob Griffiths seemed to recommend this as a long-term goal in a seminar at Perimeter Institute.

could make sense even in the microworld, but still serve as a seal of consistency wherever it is found. Specifically, one sees in a measurement a special type of correlation, and one retains only those preclusions which express such correlations. *

Not much effort has been devoted to developing this approach, unfortunately, but two potential problems have been identified, and I would like to sketch them for the benefit of the reader who is familiar with reference [6] or has thought along similar lines. The first problem, mentioned already in [6], concerns "conditioning" on the past, or on one's knowledge thereof. The question is whether one needs to introduce into the scheme a notion of "conditional preclusion" or "conditional measure", corresponding to the classical notion of conditional probability. Such a notion would express in a certain sense the "collapse of the state-vector", but it has never been clear whether it really is needed, or whether, on the contrary, all valid predictions could be made without it, on the basis of "absolute" (unconditional) preclusions. † If it turns out that it is needed, then that will represent an important lacuna in the interpretation.

The second problem, that of "specious diremptions", is harder to describe without referring to the details of the scheme elaborated in [6]. It springs from an apparent defect in the way that correlations were characterized therein, and it has the effect that essentially any event of measure zero gets re-validated as a preclusion, since it can be embedded in what counts as a correlation. Evidently, nothing would be gained by such a scheme in the way of consistency. I tend to believe that the problem is only a technical one that could be fixed by a more careful definition of correlation, but a ground for doubt is that the task is reminiscent of that of distinguishing a mere correlation between two variables from a true cause-and-effect relationship between them.

* The word "diremption" has been used for a correlation of this sort. It denotes a "splitting of the whole into two parts". Strictly speaking, however, the type of three-way correlation utilized in [6] would have to be called something like a "triremption".

† If the practitioners of the Bohmian interpretation or of the "many-worlds" interpretation are correct in their understanding of "branch formation", then a separate notion of conditional preclusion would be superfluous. However, I have never been able to convince myself that distinct "branches of the wave function" are articulated as sharply as they claim they are.

A final problem, or at least an inconvenience, is the extra work required to identify those sets of measure zero that correspond to correlations of the required type. This is reminiscent of the difficulty of identifying decohering variables in the consistent histories approach. In the context of quantum gravity, where even the causal structure is dynamical, such difficulties tend to evolve into problems of principle. Certainly, life would be simpler if one could simply affirm that all subsets of measure 0 were precluded.

Third response: a modified logic

Let us try to adhere to the original idea, according to which, without exception, the vanishing of $\mu(A)$ entails the preclusion of A. We have seen how this rule engenders contradictions of which a simple example was provided by three-slit diffraction. In that example, the slits were labeled a, b, c, and the available preclusions apparently entailed the false conclusion that the particle could never arrive at p. Imagining an instance in which the particle actually does go to p, we have in effect a contradiction between the preclusion of $A \cup B$ and the preclusion of $B \cup C$. Can we accommodate such contradictions in our logic?

Before trying to answer this question, let me introduce a notation that makes explicit a distinction which might seem to be empty, but which logicians have routinely drawn between a proposition or statement and its "truth-value". The above subset A, for example, corresponds to the proposition "The particle passes through slit a and arrives at p", but merely by referring to A, one does not necessarily assert that this proposition is true. To do that, one assigns the proposition a truth-value of 'true'. Let us agree to express this by writing $\phi(A) = 1$, with $\phi(A) = 0$ then signifying that the proposition is false. This distinction between a proposition and the assertion of the same proposition could strike one as pedantic at best, but it seems less so when one speaks in terms of the corresponding "predicates" or "properties" (the property in question in our example being that of "traversing the slit A"); and it sounds still less forced when one employs the grammar of question and answer: To the subset A then corresponds the question, *Does the particle pass through slit a (and arrive at p)?*, and to $\phi(A)$ corresponds the answer to the question, either *yes*=1 or *no*=0, as the case may be.

Expressed in terms of ϕ, our 3-slit contradiction appears as follows: $\phi(A \cup B) = 0$ and $\phi(B \cup C) = 0$, therefore $\phi(A \cup B \cup C) = 0$. Hence we could avoid the impasse if the first two equations could hold without the third holding. In question and answer form, this would amount to the following (thinking of the slits as arranged in the order (a, b, c)).

"Does the particle avoid the left two slits? Yes."

"Does the particle avoid the right two slits? Yes."

"Does the particle avoid all three slits? No."

Or if we assume that "the particle avoids a and b" always has the same truth-value as "the particle goes through c", then the first two lines become:

"Does the particle go through c? Yes."

"Does the particle go through a? Yes.",

which exhibits the contradiction still more clearly.

To embrace the contradiction is thus to accept that the particle can possess apparently incompatible attributes. But it is not altogether necessary to use this language. Instead, one can shift the emphasis and conceive of reality as represented, not by "a γ with contradictory attributes", but by ϕ itself. Whichever mode of expression one adopts, a new logic is involved, one in which the truth-value of, for example, 'X and Y' does not follow by any universal rule from the separate truth-values of 'X' and of 'Y'.

Classical logic demands that ϕ be a homomorphism between the event-Boolean algebra $\mathfrak{A}$ and the two-element Boolean algebra $\mathbb{Z}_2$ of "truth values", meaning that 'X & Y' is true if and only if 'X' and 'Y' are both true, with similar "truth tables" applying to the other logical connectives, 'or', 'not', 'xor', etcetera. Because one can prove (at least when Ω is a finite set) that any such mapping can be equated to a unique element of Ω, identifying reality with a homomorphism $\phi : \mathfrak{A} \rightarrow \mathbb{Z}_2$ ends up being nothing more than a fancy way of identifying it with an element of Ω. That is, it yields nothing new. But more general conceptions of reality arise if one modifies either ϕ or the space (of "truth values") to which it maps.[♭] In the next section, we will consider a generalization which retains the same two truth-values but relaxes the condition that ϕ be a homomorphism. Of the fundamental logical connectives, ϕ will preserve only the one known as "exclusive or".

V. An illustrative proposal: reality as a co-event

Many variations on the simple idea proposed above are possible, but all of them will describe reality by means of a particular function $\phi_{true} : \mathfrak{A} \to \mathbb{Z}_2$ which generalizes the γ_{true} of classical logic in the manner explained above. In choosing among the possible generalizations, it is useful to construe the event-algebra $\mathfrak{A}$ as literally an algebra in the strict mathematical sense of a vector space over a field, in this case the finite field $\mathbb{Z}_2$ containing only the two elements 0 and 1. To make this work, however, one must identify the algebraic sum $A+B$, not with the set-theoretic union $A \cup B$, but with what is usually called "symmetric difference", which differs from $A \cup B$ when A and B overlap:

$$A + B = (A \cup B) \backslash (A \cap B) = \{\gamma \in \Omega \,|\, \gamma \in A \cup B, \gamma \notin A \cap B\} \ .$$

With the algebraic product AB taken to be the set-theoretic intersection $A \cap B$, all the axioms for an algebra are then satisfied. For example, multiplication distributes over addition: $A(B+C) = AB + AC$, and every event A has an additive inverse $-A$ (namely A itself). Note that with these definitions $\mathfrak{A}$ becomes a *unital* algebra, its unit 1 being the whole sample space Ω, because $\Omega A = \Omega \cap A = A$. Set-theoretic complementation is then effected just by adding 1: $\Omega \backslash A = 1 + A$. (In writing things this way, I am not distinguishing between A considered as a subset of Ω and A considered as an element of the algebra $\mathfrak{A}$. This should not lead to any confusion, but it does take a little getting used to, especially if one further equates A to the corresponding predicate. One then has three distinct sets of notation for essentially the same set of operations: the set-theoretical notations like "$\cap$", the algebraic notations like "+" and the logical notations like "&"!)

[♭] Logics that generalize $\mathbb{Z}_2$ seem to be called "modal". Such modifications have been proposed by many authors, notably in [7]. "Quantum logic" in the original sense of these words seems to have consisted primarily in replacing $\mathfrak{A}$ by the lattice of subspaces of a hilbert space. It did not seem to put forward any definite proposal for ϕ, concerning itself more with probabilities than truth-values. The emphasis herein will fall neither on $\mathfrak{A}$ nor on $\mathbb{Z}_2$, but on the character of the mapping between them.

That ϕ classically is a unital algebra homomorphism signifies that it then preserves sum, product and unit; that is

$$\phi(A+B) = \phi(A) + \phi(B)$$
$$\phi(AB) = \phi(A)\phi(B)$$
$$\phi(1) = 1$$

I will call the first condition linearity, and the second multiplicativity. Which of these conditions shall we retain — if any — and which shall we modify? If we were to drop all of them, the resulting framework would be essentially vacuous. If we were to keep them all, we'd be back to classical logic. We've already narrowed down the field of possibilities by choosing to retain $\mathbb{Z}_2$ as the "codomain" of ϕ (rather than generalizing it to, say, a finite field of characteristic two). With this choice, $\phi(A)$ can be only 1 (true) or 0 (false). If we want to preserve the symmetry between subsets and their complements (that is, if we don't want to be forced to decide for each predicate, whether to regard it as "positive" or "negative" in nature), then it's natural to assume that $\phi(1+A) = 1 + \phi(A)$, which says that a statement is true iff its negation is false. But this is a special case of linearity, as long as ϕ is unital.[⋆] It thus seems natural to make the following choice: retain linearity and drop multiplicativity. It also seems natural to retain the condition that ϕ be unital, but we will lose nothing by leaving that choice open for now.

Let us agree to call a linear function $\phi : \mathfrak{A} \to \mathbb{Z}_2$ a *co-event*, and to qualify ϕ as *preclusive* if it assigns 0 to every $A \subseteq \Omega$ of μ-measure zero. Reality, then, is supposed to be a preclusive co-event.[†]

⋆ If ϕ is not unital, then necessarily $\phi(1) = 0$. In this case linearity implies that a statement is true iff its negation is *true*! In either case, the truth of a statement is determined by that of its opposite, preserving the desired symmetry.

† A linear mapping from $\mathfrak{A}$ to the space of scalars $\mathbb{Z}_2$ defines an element of the vector space dual to $\mathfrak{A}$. Hence the nomenclature "co-event". In terms of logical operations, the linearity of ϕ means that the truth-value of (A XOR B) — its truth or falsehood — still follows from the separate truth-values of A and of B, even though — because we have dropped multiplicativity — the truth-value of (A AND B) does not. (Here 'XOR' is exclusive or).

Before considering what further conditions we might want our co-events to satisfy, let us see how linearity and preclusivity work out in the three-slit example described earlier. Since the subsets A, B and C are mutually disjoint, their unions can be written as simple sums. Then, because ϕ has been assumed to be preclusive, we have as before (but with our new notation): $\mu(A+B) = 0$ implies $\phi(A+B) = 0$, and $\mu(B+C) = 0$ implies $\phi(B+C) = 0$. The deductive train leading to a contradiction, now gets switched to a new track, however, because $\phi(A+B) = 0$ and $\phi(B+C) = 0$ no longer combine to imply $\phi(A+B+C) = 0$. Instead they imply only that $\phi(A+C) = 0$, as proven by the brief computation, $\phi(A+C) = \phi(A+B+B+C) = \phi(A+B) + \phi(B+C) = 0+0 = 0$, where I've used that $B+B = 0$ and $\phi(0) = 0$ always.[♭] If we assume further that the particle really does arrive at p, then we have additionally $\phi(A+B+C) = 1$, which implies in the same way that $\phi(A) = \phi(B) = \phi(C) = 1$. This, then, is the unique preclusive co-event ϕ for which the particle arrives at p. In particular there *is* such a co-event, unlike with classical reasoning, which drove us to conclude that $\phi(A+B+C) = 0$. In this way, we have avoided the false prediction.

In the 3-slit example, the relevant Boolean algebra $\mathfrak{A}$ is finite dimensional, which, over the finite field $\mathbb{Z}_2$, implies that $\mathfrak{A}$ itself is a finite set. Whenever this happens, one may conveniently represent a co-event as a subset of Ω. In order to see this, note first that[⋆] any history $\gamma \in \Omega$ defines a (unital) co-event γ^* through the equation $\gamma^*(A) = \Theta(\gamma \in A)$, where the notation $\Theta(\gamma \in A)$ denotes 1 if $\gamma \in A$ and 0 if $\gamma \notin A$. Notice further that we can suppose without loss of generality that Ω itself is finite, and then the γ^* furnish a canonical basis for $\mathfrak{A}$ regarded as a vector space over $\mathbb{Z}_2$. This means that any coevent ϕ can be expanded in terms of the γ^*, and we may therefore identify it with the terms that occur nontrivially in the expansion. In this limited sense a co-event is nothing but a subset of the sample space (a subset of odd cardinality if we require that ϕ be unital).

♭ The earlier deduction implicitly used multiplicativity, which, had we retained it, would yield $\phi((A+B)(B+C)) = 0$, and thence $\phi(B) = 0$ since $(A+B)(B+C) = AB + AC + BB + BC = 0 + 0 + B + 0 = B$. From this follows $\phi(A) = \phi(A+B) = 0$, $\phi(C) = 0$, etc.

⋆ If the singleton set $\{\gamma\}$ belongs to $\mathfrak{A}$ (as it normally will when $|\Omega| < \infty$) then it is an *atom* of $\mathfrak{A}$. More generally, any atom $x \in \mathfrak{A}$ defines a unital co-event x^* by $x^*(y) = \Theta(xy = x)$. (Like γ^*, such an x^* is actually a homomorphism.)

In general, however, Ω will be infinite, and the co-events will not be expressible so simply (although finite subsets of Ω will still yield co-events). In the 3-slit example, our unique co-event was just $\phi = A^* + B^* + C^*$ (where the notation presupposes that we've limited $\mathfrak{A}$ to the algebra generated by A, B and C.)

Our three-slit example illustrates how one may avoid (or perhaps I should say accommodate) the antinomies engendered by combining preclusion with classical logic, but it is rather a rudimentary example, and it certainly does not prove that things will always work out so nicely. Lacking a general theorem to that effect, we can only try the basic ideas out on other examples of interest. Let us therefore work out one example more, namely the EPRB-like gedankenexperiment introduced by Lucien Hardy, which illustrates almost every known interpretive difficulty.

Before taking up this example, however, we must decide how to deal with the fact that not every preclusive co-event is really suitable as a potential reality, but only those which, in a sense still to be fully determined, are "elementary". Perhaps the simplest way to recognize the problem is to return to classical probability, but with the measure μ regarded now as a special case of a quantal measure, to which our general scheme can therefore be applied. Classically, one may encounter sets of measure zero, but never interference, and consequently every subset of a precluded subset is also precluded. From this, it is not hard to conclude (once again taking Ω to be finite) that any γ whose measure is itself nonzero yields the preclusive co-event γ^*, and the most general preclusive co-event is an arbitrary sum of these. On the other hand, we know that the "right answer" in this case (we are just doing classical physics!) is that reality should be identified with a *single* γ, not with a set of them. The difficulty arises because we have abandoned multiplicativity, consequently allowing arbitrary sums of preclusive events to survive, since they clearly are also preclusive. But this surely is not appropriate. (For example the sum could include co-events that differed macroscopically from each other.) What we seem to want to do is to reduce such sums to their "elementary parts" and keep only the latter on our roster of potential realities. That is, we want to allow multiple terms in the sum only when they are mandated by quantal interferences, not otherwise. One way to do this is by *ordering* the co-events appropriately and keeping only the minimal ones.

To that end, let me define the *support* of a co-event as the set of γ that comprise it when we regard it as a subset of Ω. This definition works when Ω is finite, but not otherwise. A more general scheme is suggested in the appendix. Anyhow, with this definition, let us say that $\phi_1 \prec \phi_2$ when $\mathrm{supp}(\phi_1) \subseteq \mathrm{supp}(\phi_2)$; that is, we attribute to the co-events the order induced by set-theoretic inclusion on the subsets of Ω to which they correspond. (For example, $A^* + C^* \prec A^* + B^* + C^*$.) We can then limit our potential realities to those preclusive co-events which are *minimal* among all preclusive co-events. In the classical situation we were just considering, this yields exactly the right answer, that only the "atomic" coevents of the form $\phi = \gamma^*$ are retained. In other words, we have found a condition which is indistinguishable from multiplicativity in the classical case, but which will allow the kind of more general co-event that we need for the quantal case. (For example, our 3-slit co-event $\phi = A^* + B^* + C^*$ is certainly minimal, since no other preclusive co-events exist at all.)

We are now prepared to take up the Hardy Gedankenexperiment [8]. In a histories setting it is convenient to conceive of this experiment in terms of the trajectories of a correlated pair of spin-$\frac{1}{2}$ silver atoms passing through successive Stern-Gerlach beam-splitters followed by recombiners. If the atom on the left (say) passes first through a σ_z-analyzer, then a recombiner, and then a σ_x-analyzer, it can follow any of four possible trajectories that we can label with a pair of $\pm$ signs, and similarly for the atom on the right. In all, we have then a sample space Ω of $4 \times 4 = 16$ members which we can label $[+ + ++]$,...,$[- - --]$, taking the signs in the order: z-left, z-right, x-left, x-right. For example, $[+ + --]$ is the trajectory such that both atoms traverse the "upper" beams in their respective z-analyzers, and subsequently the "lower" beams in their respective x-analyzers. (Compare the description in [9].) We further imagine an idealized source that emits the atoms initially, with equal amplitudes for each of the pairs $(\sigma_z^{left}, \sigma_z^{right}) =$ $(++)$, $(+-)$, $(-+)$, but with zero amplitude for the pair $(--)$. The decoherence functional D and quantal-measure μ are then determined as usual. (See [9].)

The sixteen elements of Ω produce a Boolean algebra $\mathfrak{A}$ of $2^{16} = 65536$ events, of which a typical example, consisting of three trajectories, is[†]

$$[++--]+[+---]+[-+--]\ . \tag{4}$$

Now this particular event is important because it turns out to be precluded classically but not quantally (like the subset $A+B+C$ in our three-slit example). Unfortunately there is no space here to review why this fact is interesting for "hidden variables" theories, nor is there space even to do justice to the many interesting mathematical and causal aspects of the gedankenexperiment as an example of quantal measure theory. Instead I will just enumerate the results and leave their further analysis for another time.

By "results" I mean here the determination of the measure-zero events and of the resulting minimal preclusive co-events. The former (or at least all that I know of) turn out to be the following (see added note 2):

$$[--xy]\ ,$$

$$[++x-]+[+-x-], \quad [++-x]+[-+-x]\ ,$$

where each 'x' or 'y' may be replaced by either '+' or '$-$'. (In addition, as follows from the lemmas of §II, any A which is a disjoint sum of these also satisfies $\mu(A)=0$, but by linearity that adds nothing to the determination of the preclusive co-events.) The preclusive co-events are those that map these eight precluded events to $0 \in \mathbb{Z}_2$. Among them are eight minimal ones, namely

$$[++++]^*$$

[†] For the sake of notational simplicity, I will not distinguish between an element γ of Ω and the corresponding singleton subset $\{\gamma\}$, which is an element of $\mathfrak{A}$ but not strictly speaking of Ω. Thus, for example, (4) denotes the element of $\mathfrak{A}$ given by the set whose members are the three trajectories, $[++--]$, $[+---]$ and $[-+--]$. (An object like $[+--+]$ is not realistically an element of the full physical sample space anyway, because it really comprises several trajectories and because there is more in the world than just a pair of silver atoms. In a formalization that took this into account, our 65536 events would comprise a proper subalgebra of the full physical Boolean algebra; $[+--+]$ would be an atom of this subalgebra; and sample-spaces as such would hardly need to be be mentioned.)

$$[+-++]^* \qquad [-+++]^*$$

$$[-++-]^* \qquad [+--+]^*$$

$$[++--]^* + [+---]^* + [-+--]^*$$

$$[+++-]^* + [+-+-]^* \qquad [++-+]^* + [-+-+]^*$$

Of these, the first five are "classical" (belonging to a single element of Ω), and the last two fail to be unital, since they contain an even number of summands. The sixth is thus the unique minimal preclusive co-event which remains unital while going beyond classical logic. The most important thing to notice is that every "observable event" is assigned 'true' by some unital co-event in the list, where by an "observable" event I mean one that would be seen with nonzero probability by an external agent who decided to look for it. In particular, the event "both atoms in the lower beams after the second splitting" occurs when ϕ_{true} is the sixth co-event in the list. In this sense our scheme has added to the classically possible co-events only the bare minimum needed to be compatible with the experimental facts, as predicted by standard quantum mechanics.

VI. Further work and final comments

The thesis of this paper has been that a quantum theory should admit a self-contained formulation based first of all on a space Ω of "generalized trajectories" or "histories", and secondly on a "quantal measure" μ defined on Ω. In particular, I argued that new interpretive possibilities arise in such a framework which promise to provide a more satisfactory concept of "quantal reality" than offered by either "the ensemble of instrument readings" or "the wave-function ψ" (with or without "spontaneous collapse"), or by "a single element of Ω". To illustrate this contention, I proposed to situate reality in a minimal preclusive co-event as defined in section V, which is a "truth-function" ϕ mapping $\mathfrak{A}$ to $\mathbb{Z}_2$, where $\mathfrak{A}$, the domain of μ, is a Boolean algebra of "events" (subsets of Ω), and ϕ is linear when addition is interpreted as the so called symmetric-difference of subsets (corresponding to the logical connective, exclusive-or). The "dynamical laws" governing ϕ flowed from a condition of "preclusivity", which required that $\phi(A) = 0$ for any $A \in \mathfrak{A}$ such that $\mu(A) = 0$. To help clarify the meaning of the proposal, we worked out two examples, involving 3-slit diffraction and EPRB-like correlations.

However two important questions have been addressed only in the most implicit manner. The first question is "How does the past condition the future?", which gets at the formal basis of prediction in the scheme; the second is "How does probability get into the story?". Although a full answer to either question could of course be given only in the course of an extensive further working out of the basic ideas, sketches of the answers are already implicit in the basic assumptions of the scheme, as set out above.

The answer to the first question resembles the answer that the Bohmian scheme would have to give: a past reality sets a "boundary condition" for future realities because the future must prolong the past. The set of possible events codified in $\mathfrak{A}$ grows with time in the sense that $\mathfrak{A}(t_1) \subset \mathfrak{A}(t_2)$ when $t_1 < t_2$. Conversely any coevent ϕ_2 for $\mathfrak{A}(t_2)$ induces by restriction a coevent for $\mathfrak{A}(t_1)$, and this relationship entails a condition that ϕ_2 must satisfy, once ϕ_1 is given. It is this consistency condition that expresses the influence of the past on the future.[♭] More generally (in quantum gravity) "the past" might no longer correspond to a simple subalgebra of $\mathfrak{A}$, but information about it would still limit the possible ϕ's in an analogous manner e.g. by specifying that certain "stem-events" [10] occurred.

The answer to the second question is, as indicated earlier, that probability doesn't "get in" at all; it's supposed to be there already in the form of the "preclusion rule", as applied to events A of sufficiently small measure $\mu(A)$. In an "actual ensemble" of repetitions of the same experiment, outcomes with the wrong frequencies will (we hope) be precluded. (Notice that this is not meant to introduce some classical probability measure on the space of co-events. Such probabilities as are meaningful are supposed to follow from the *quantal* measure μ on Ω, together the preclusion rule and the interpretation of reality as a co-event.) It remains to be seen whether one can capture all valid statistical predictions in this way.

Let me close with a list of open questions, some relating to the new notion of quantal reality proposed above, and some relating to "histories quantum mechanics" more generally.

♭ Subtle and important issues arise here, in connection with the *minimality* conditions we have imposed on ϕ; for minimality is in general not "stable under time-development". One can find good examples of this in the second gedankenexperiment of section V.

- Will we be forced to accept co-events ϕ which assign 'true' to two or more macroscopically distinct alternative events? (this being perhaps the greatest danger for the scheme of §V, assuming no simpler inconsistencies show up).

- Will the "mutability of minimality" lead to trouble with causality? (this being perhaps the second greatest danger.)

- How can one (or should one) incorporate fermionic (Grassmann) variables into a histories formulation of quantum field theory?

- Can the interpretive scheme of section V shed any light on whether field-particle duality has a home in histories-based quantum field theory? (Perhaps it opens up a narrow window for this.)

- Does the quantal random walk defined in [11] have, as conjectured, a continuum limit that reproduces the quantum mechanics of the free non-relativistic particle? (This random walk appears to offer one of the few fully defined examples of a genuinely non-unitary quantal process.)

- How does relativistic causality manifest itself in a histories framework? Does there exist a suitable notion of "quantum screening off", as essayed in [9]?

- Would anyone care to do the three-slit null test of quantum mechanics described in section II above and in references [3] and [6]?

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Appendix. Definition of the order $\prec$ in the general case

Given two coevents $\phi_{1,2} : \mathfrak{A} \to \mathbb{Z}_2$, we defined $\phi_1 \prec \phi_2$ in the body of the paper by $\operatorname{supp}\phi_1 \subseteq \operatorname{supp}\phi_2$. Although an analogous criterion can be formulated for infinite Boolean algebras, it doesn't seem to be what we want. Instead the following appears to be the appropriate generalization from finite to infinite Boolean algebras:

$$\phi_1 \prec \phi_2 \text{ iff } (\exists S \in \mathfrak{A})(\forall A \in \mathfrak{A}) \ \phi_1(A) = \phi_2(SA) \ .$$

It is not difficult to see that $\prec$ thus defined is reflexive, transitive and acyclic, and that it agrees with the definition in the main text when $\mathfrak{A}$ is finite.

Added note 1: Later work by Yousef Ghazi-Tabatabai shows that the linearity condition of §V is too restrictive to accommodate certain examples, the simplest of which involves 4-slit diffraction.

Added note 2: Cohl Furey has now exhaustively classified the precluded subsets for the gedankenexperiment of §V, discovering two more independent ones, in addition to those listed above, namely $[+-+-]+[-++-]$ and its symmetric twin, $[-+-+]+[+--+]$. In consequence, the number of minimal preclusive (linear) coevents is reduced to six. Of the eight listed above, the first three and the sixth survive as is, but the fourth and seventh must be added together, as also the fifth and eighth. All six of the resulting coevents are unital.

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