Last time

We defined the concept of a "covariant derivative".

We argued that if $\Gamma^\mu_{\nu\rho}$ transforms non-tensorially

$$\nabla_{\nu} \xi^\mu = \partial_{\nu} \xi^\mu + \Gamma^\mu_{\nu\rho} \xi^\rho$$

defines a covariant derivative on vector fields.

Property IV) of a covariant derivative,

$$\nabla_{\nu} (f) = \nabla_{\nu} f = \delta^\mu_{\nu} \nabla_{\mu} f$$

allows us to find how the covariant derivative acts on scalar functions.
\[ \nabla (f) = \nabla \cdot \varepsilon \cdot f = \nabla \cdot \varepsilon \cdot f \]
\[ = \nabla \cdot \varepsilon \cdot f \]
\[ = \nabla \cdot (\varepsilon \cdot f) \]
\[ = \varepsilon \cdot \nabla f \]

Exercise 15

Using the last relation, show that the covariant derivative of a form has components

\[ \nabla \omega \mu = \varepsilon \omega \mu - \Pi_{\nu \mu} \omega \nu \]

The covariant derivative of an arbitrary tensor is then

\[ \nabla \mu T_{\nu} \rho = \varepsilon T_{\nu} \rho + \Pi_{\nu \sigma \mu} T_{\sigma \rho} - \Pi_{\sigma \mu} T_{\nu \rho} \]

and so on...

(e.g. \( \nabla \mu \delta_{\rho} \beta = 0 \)).
There are as many covariant derivatives as connections.

In the presence of a metric, a natural condition to impose is that the covariant derivative should commute with

\[ \nabla_x (g_{uv} v^s) = g_{uv} \nabla_x v^s \quad \forall v^s. \]

Using Leibniz' rule:

\[ (\nabla_x g_{uv}) v^s + g_{uv} \nabla_x v^s = g_{uv} \nabla_x v^s, \]

or

\[ \nabla_x g_{uv} = 0 \quad \text{(metric compatibility)} \]

This requirement does not suffice to determine the connection:
Suppose we are given an arbitrary "tornion" tensor $T^\mu{}_{\nu\rho} = -T^\mu{}_{\rho\nu}$, and a metric-compatible connection $\Gamma^\mu{}_{\nu\rho}$.

Define a new connection by

$$\tilde{\Gamma}^\mu{}_{\nu\rho} = \Gamma^\mu{}_{\nu\rho} + \frac{1}{2} \left( T^\mu{}_{\nu\rho} - T^\mu{}_{\rho\nu} + T^\mu{}_{\rho\nu} \right)$$

Exercise 16

Show that the covariant derivative $\tilde{\nabla}^\mu$ linked to the new connection $\tilde{\nabla}^\mu$ also satisfies $\tilde{\nabla}^\mu g_{\nu\rho} = 0$.

Exercise 17

Show that $\Delta \tilde{\Gamma}^\mu{}_{\nu\rho}$ is antisymmetric,

$$\Delta \tilde{\Gamma}^\mu{}_{\nu\rho} = -\Delta \tilde{\Gamma}^\mu{}_{\rho\nu}$$
any metric-compatible connection $\tilde{\nabla}$ can be written as above, with $\Gamma^m_{uv} = \Gamma^m_{vu}$.

**Exercise 18**

Show that $\tilde{\nabla}_u \tilde{\nabla}_v f - \tilde{\nabla}_v \tilde{\nabla}_u f = -T^m_{\mu v} \tilde{\nabla}_m f$

with torsion $T^m_{\mu v}$.

A metric-compatible connection would require additional structure (the torsion tensor). Theories of this kind are known as "Einstein-Cartan" theories.

In GR we postulate vanishing torsion:

$T^m_{\mu v} = 0$.

**Exercise 19**

Show that the connection of a metric-compatible covariant derivative with vanishing torsion ($\Gamma^m_{uv} = \Gamma^m_{vu}$)
is given by
\[
\nabla_a \mu = \frac{1}{2} \nabla_{(a} \nu_{\beta \gamma \delta \epsilon \lambda \mu \alpha \beta \gamma \delta \epsilon \lambda \mu)} - \frac{1}{2} \nabla_{\nu} \frac{\partial \nu}{\partial x^\alpha} + \frac{\partial \nu}{\partial x^\alpha} \nabla_{\nu}
\]

This is known as the Christoffel connection. The \( \Gamma \)'s are also known as the Christoffel symbols.

The Christoffel symbols satisfy a number of useful identities:

Exercise 20

Show that \( \Gamma^\nu_{\mu \nu} = \frac{1}{\sqrt{\det g}} \partial_\nu (\sqrt{\det g}) \)

As a consequence \( \int dV \; \partial_\nu \Gamma^\nu_{\mu \nu} = \int dV \; (\partial_\nu \Gamma^\nu_{\mu \nu}) \)

is a boundary term. We can therefore integrate by parts
\[
\int dV \; \partial_\nu \nabla^\nu = -\int dV \; (\nabla^\nu T \nabla_\nu T) \cdot \nabla.
\]
other identities:

\[ \frac{\partial g_{\mu \nu}}{\partial x^\sigma} = g_{\mu \sigma} \Gamma^\sigma_{\nu \rho} + g_{\nu \sigma} \Gamma^\sigma_{\mu \rho} \]

\[ \frac{\partial g_{\mu \nu}}{\partial x^\sigma} = -g_{\nu \sigma} \Gamma^\sigma_{\mu \rho} - g_{\mu \sigma} \Gamma^\sigma_{\nu \rho} \]

geodesics

The covariant derivative tells us how a tensor \( T \) changes along a direction \( \mathbf{u} \):

\[ \nabla_\mathbf{u} T = \nabla^\mu \nabla_\mu T. \]

If a vector \( \mathbf{w} \) is parallel-transported along \( \mathbf{u} \), its covariant derivative vanishes:

\[ \nabla_\mathbf{u} \mathbf{w} = 0 \]

Definition

A geodesic is a curve that parallel-transports its own tangent vector:

\[ \nabla^\mu \mathbf{v}_\mu = 0. \]
Because the components of the tangent vector to a curve \( \gamma(\lambda) \) are
\[
\gamma^{\nu} = \frac{dx^{\nu}}{d\lambda} , \quad \text{we find}
\]
\[
\frac{d x^{\nu}}{d \lambda} \left[ \frac{\partial \gamma^{\nu}}{\partial x^{\mu}} + \Gamma_{\mu \rho}^{\nu} \frac{dx^{\rho}}{d\lambda} \right] = 0
\]
\[
\frac{d^2 x^{\nu}}{d \lambda^2} + \Gamma_{\mu \rho}^{\nu} \frac{dx^{\mu}}{d\lambda} \frac{dx^{\rho}}{d\lambda} = 0
\]

Geodesic equation

Note that to solve the geodesic equation, we need to specify \( x^{\nu}(\lambda_0) \) (position)
\[
\frac{dx^{\nu}}{d\lambda}(\lambda) \quad (\text{"speed")}
\]

Curves that satisfy the geodesic equation are "geodesics". In general relativity, time-like neutral test bodies move along geodesics:
\[
\gamma_{\mu \nu} \frac{dx^{\mu}}{d\lambda} \frac{dx^{\nu}}{d\lambda} < 0
\]
We could have derived the geodesic equation from the equivalence principle:

In a freely falling frame, neutral test bodies move along straight lines:

\[
\frac{d^2 x^u}{d\lambda^2} = 0 \quad \text{\( R^u_{\alpha\beta}(\rho) = 0 \)}
\]

The geodesic equation is the generally covariant generalization of this equation, valid in an arbitrary coordinate system.

Thus, geodesics are locally "straight".

Note that along a geodesic

\[
\nabla^u (g_{\rho\sigma} u^\rho u^\sigma) = 0
\]

\[
\frac{dx^u}{d\lambda} \partial_\mu (g_{\rho\sigma} u^\rho u^\sigma) = \frac{d}{d\lambda} \left( g_{\rho\sigma} u^\rho u^\sigma \right) = \frac{d}{d\lambda} \left( g_{\rho\sigma} \frac{dx^\rho}{d\lambda} \frac{dx^\sigma}{d\lambda} \right)
\]

Thus, for a geodesic
\[
\frac{d}{d\gamma} \left[ \left( \frac{ds}{d\gamma} \right)^2 \right] = - \frac{d}{d\gamma} \left[ \left( \frac{d\tau}{d\gamma} \right)^2 \right] = 0. 
\]

Proper time along a geodesic is hence a linear function of \( \gamma \). We can therefore choose \( \gamma \) to be proper time:

\[
\frac{d\tau}{d\gamma} = 1 \implies \left( \frac{d\tau}{d\gamma} \right)^2 = 1 \implies g_{\mu\nu} \nu^\mu \nu^\nu = -1
\]

Example

In Minkowski spacetime, the geodesic equation is \( \frac{d^2 x^\mu}{d\tau^2} = 0 \). Therefore, \( x^\mu \) is a linear function of \( \tau \). The condition \( g_{\mu\nu} \nu^\mu \nu^\nu = -1 \) implies

\[
\gamma_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} = - \left( \frac{dt}{d\tau} \right)^2 + \left( \frac{dx^\mu}{d\tau} \right)^2 = -1
\]

\[
-(\frac{dt}{d\tau})^2 + \left( \frac{dx^\mu}{d\tau} \right)^2 = -1 \quad \left( \frac{dt}{d\tau} \right)^2 [1 - \dot{x}^\mu \dot{x}^\mu] = 1
\]
or \[
\frac{d\tau}{dt} = \sqrt{\frac{1}{1-v^2}}
\]

If a particle moves at speed \( v \), a time interval \( d\tau \) in the rest frame of the particle seems to last \( d\tau = \frac{dt}{\sqrt{1-v^2}} \) for an observer at \( u \)t = \( \tau \) time dilation.

**Exercise 21**

Show that the geodesic equation follows by extremizing the proper time along a timelike curve (with fixed end points):

\[
\tau = \int \sqrt{-g_{\mu \nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}} \, d\lambda = \tau [X^\mu(\lambda)]
\]

**Hint:** \( \tau \) is parameterization independent. Simplify your life by imposing the "gauge condition"

\[
g_{\mu \nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} = -1
\]