Last time

The metric is a symmetric, non-degenerate
rank $(0,2)$ tensor

\[ ds^2 = g_{\mu\nu} dx^\mu dx^\nu. \]

Example

Minkowski space is the spacetime of
special relativity. Its metric is

\[ ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu, \quad \eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \]

The vectors $e^\mu$ form an orthonormal
basis with $ds^2 (e^\mu, e^\nu) = \eta_{\mu\nu}$. Minkowski spacetime has Lorentzian signature.
In an arbitrary spacetime, we cannot find coordinates in which the spacetime metric is that of Minkowski. However it is always possible to choose such that at a single, arbitrary point \( p \),

\[
g_{ab}(p) = \eta_{ab}; \quad \frac{\partial g_{ab}}{\partial x^\sigma} \bigg|_p = 0.
\]

(See e.g. Carroll for a proof).

In such coordinates spacetime is manifestly "locally flat." These coordinates correspond to those of a freely falling observer. In this "local inertial frame" the effects of gravity have disappeared, since locally spacetime appears to be that of special relativity.
The metric allows us to find the length of a curve \( \gamma(\lambda) \mapsto x^m(\lambda) \).

Suppose the vector \( \dot{\gamma} \) associated to the curve is "space-like" at each point in \( t \):

\[
g(\dot{\gamma}, \dot{\gamma}) > 0.
\]

Recall that the components of \( \dot{\gamma} \) are

\[
\dot{x}^m = \frac{dx^m}{d\lambda} = \frac{dx^m}{d\alpha}.
\]

Hence,

\[
g(\dot{\gamma}, \dot{\gamma}) = g_{mn} \frac{dx^m}{d\lambda} \frac{dx^n}{d\lambda} = \frac{ds^2}{d\lambda^2}.
\]

We hence define

\[
\begin{align*}
\mathcal{L} &= \int ds = \int d\lambda \sqrt{g_{mn} \frac{dx^m}{d\lambda} \frac{dx^n}{d\lambda}} \\
&> 0 \text{ by assumption}
\end{align*}
\]
If the curve is "timelike", \( g(\sigma, \tau) < 0 \), we define instead the proper time along the curve to be

\[
T = \int dt = \int ds \sqrt{-g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}}
\]

**Exercise 12**

Show that the length (or proper time) along a curve is parameterization-independent.

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**Integration**

One can show that, given a metric, there is a natural way to integrate a function over a manifold.

[In reality, one defines the integral of an antisymmetric rank \((n,0)\) tensor.]

\[
\int f \, dV = \int d\sigma \ldots d\sigma \sqrt{|\det g_{\mu\nu}|} \ f(x)
\]
Exercise 13

Show that the "volume element"
\[ dV = dx_1 \ldots dx_n \sqrt{|\det g_{ij}|} \]

is invariant under coordinate transformations.

Nota: \( g_{ij} \) stands for the matrix with elements \( g_{rin} \).

Corollary

\[ \int_M dV \cdot f \quad \text{is well-defined. Its value does not depend on the coordinate system.} \]

Exercise 14 (Non-Euclidean geometry)

The metric of the Poincaré disc is
\[ ds^2 = d\theta^2 + \sinh^2 \theta \, ds^2. \]

i) Calculate the length of a circle at
\[ \theta = \Theta_0 = \text{const}. \]
ii) calculate the length of a line from $\theta = 0$ to $\theta = \theta_0$ at $g = \text{const}$. 

iii) calculate the area of the region with $0 \leq \theta \leq \theta_0$ (as a function of "distance" to $\theta = 0$.)

iv) compare your results to those on the surface of a sphere, 

$$ds^2 = d\theta^2 + \sin^2 \theta \, d\phi^2.$$ 

Raising and Lowering Indices

If we have a metric, there is a natural way to identify vectors and forms:

Given a vector $\mathbf{v}$, we can define a form $\omega$ related to $\mathbf{v}$ by

$$\omega(x) = g(x, \mathbf{v}) \quad \forall x \in \mathbb{M}.$$
In components:

$$w^\mu = g^\mu_\nu v^\nu \quad \forall t = 1$$

$$\Rightarrow \quad w^\mu = g^\mu_\nu v^\nu.$$

In practice we drop the distinction between $e$ and $n$, and simply write

$$v^\mu = g^\mu_\nu v^\nu.$$

We have "lowered the index" with the metric.

Because the metric is non-degenerate, the map between vectors and forms defined above is invertible. The inverse defines a map from forms to vectors. Let its components be $(g^{-1})^\mu_\nu$.

Then, by definition

$$(g^{-1})^\mu_\nu g_\nu^\rho = \delta^\mu_\rho,$$

we drop the inverse and simply write
\[ g^\mu \cdot g_{\nu} = \delta^\mu_\nu. \]

We can use the rank (2,0) tensor \( g^\mu \), to map vectors to forms:

\[ w^\mu = g_{\mu \nu} w^\nu. \]

We "kiss" indices with the inverse metric.

We can use both maps \( g^\mu \) and \( g_\mu \)
to convert between covariant and contravariant slots in a rank + (k, c) tensor. E.g.

\[ T_{\mu \nu} = g_{\mu \alpha} g_{\nu \beta} T^{\alpha \beta}. \]

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2.6. Covariant derivatives

But so far, we still do not know how to compare vectors (or tensors) at different points of the manifold, or to figure out if two vectors are different.
Definition

A covariant derivative operator $\nabla$ on a manifold is a map from the space of rank $(k, l)$ tensors to the space of rank $(k, l+1)$ tensors satisfying:

i) Linearity: $\nabla(T + \beta S) = \alpha \nabla T + \beta \nabla S$

ii) Leibniz rule: $\nabla(T \otimes S) = (\nabla T) \otimes S + T \otimes (\nabla S)$

iii) Commutes with contraction + $T \otimes (\nabla S)$

1) Consistency with the union of tangent vectors:

\[
\nabla(f) = \nabla_\xi f
\]

We can define a derivative operator on vector fields by the following prescription:

\[
(\nabla \nabla)^\mu_\nu = \nabla_\mu \nabla^\nu = \partial_\mu \nabla^\nu + \Gamma^\nu_{\mu\lambda} \nabla^\lambda
\]
This transforms as a tensor only if 

\[ \Pi^\mu_{\rho\sigma} \ \text{transforms non-tensorially} \]

(we shall not need to know how it transforms)

\( \Pi \) (the connection) tells us how the vector components change if we transport the vector from \( x \) to \( x + \Delta x \)

\[ ( \nabla \nu \ S^\mu ) = \frac{\nabla^\mu (x + \Delta x) - \nabla^\mu (x + \Delta x)}{\Delta x^\nu} \]

\[ = \frac{\nabla^\mu (x + \Delta x) - [\nabla^\mu (x) - \Gamma^\mu_{\rho\nu} \ n^\rho (x) \Delta x^\nu]}{\Delta x^\nu} \]

\[ = \partial_\nu \ S^\mu + \Gamma^\mu_{\rho\nu} \ n^\rho \]
Condition (iii) allows us to define
the covariant derivative of a form \( \pi \):

\[
\nabla (\pi) = \nabla \cdot \pi \cdot \pi (f) = \nabla \cdot \nabla \cdot f = \nabla \cdot \nabla \cdot f
\]

\[
\Rightarrow \quad \nabla \cdot f = \nabla \cdot f.
\]

**Exercise 15**

Using the last relation, show that the covariant derivative of a form has components

\[
\nabla_{\mu} \omega_\nu = \nabla_\mu \omega_\nu - \Gamma^\rho_{\mu \nu} \omega_\rho
\]

There are as many covariant derivatives as connections \( \nabla \).