Recap

- Vectors are "derivative operators" that act on functions $f : M \to \mathbb{R}$
- Forms are linear maps acting on vectors
- Tensors are multilinear maps acting on forms and vectors

Remark

There is a natural way to identify $(T_p M)^{**}$ and $T_p M$:

We identify $\omega^{**} \in (T_p M)^{**}$ and $\omega \in T_p M$ by the condition

$$\omega^{**}(\omega) = \frac{1}{n} \omega(\omega).$$
given a set of vectors $v_1, \ldots, v_k$ and
forms $w^1, \ldots, w^k$, we can construct a
rank $(k,l)$ tensor by tensor product:

$$ T = v_1 \otimes \cdots \otimes v_k \otimes w^1 \otimes \cdots \otimes w^k $$

$T$ acts on a set of $k$ forms $\omega^1, \ldots, \omega^k$ and $l$ vectors $x^1, \ldots, x^l$ as

$$ T(\omega^1, \ldots, \omega^k; x^1, \ldots, x^l) = \omega^1(x^1) \cdots \omega^k(x^k) \omega^1(\omega^1) \cdots \omega^k(\omega^k) $$

(Such a map is multilinear by construction).

**Exercise 8**

i) show that the tensors

$$ e^\mu_1 \otimes \cdots \otimes e^\mu_k \otimes dx^{\nu_1} \otimes \cdots \otimes dx^{\nu_l} $$

$(n^k \times n^l)$ of them) form a basis
of the space $T^{(k,l)}_p M$. 

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ii) Show that the components on any rank \((k,l)\) tensor \(T\) in this basis are

\[ T^{\mu_1 \cdots \mu_k} v_1 \cdots v_l = T (dx^{\mu_1}, \ldots, dx^{\mu_k}; e_{v_1}, \ldots, e_{v_l}), \]

that is

\[ T = T^{\mu_1 \cdots \mu_k} v_1 \cdots v_l \epsilon_{\mu_1} \cdots \epsilon_{\mu_k} \otimes dx^{v_1} \cdots \otimes dx^{v_l}. \]

iii) Show that under a change of coordinates

\[ T'^{\mu_1 \cdots \mu_k} v_1 \cdots v_l = \frac{\partial x^\mu}{\partial x'^{\mu'}} \cdots \frac{\partial x^\mu}{\partial x'^{\mu_k}} \frac{\partial x^\nu}{\partial x'^{\nu_1}} \cdots \frac{\partial x^\nu}{\partial x'^{\nu_l}} T^{\mu'_1 \cdots \mu'_k} v_1 \cdots v_l \]

iv) Show that the components of the tensor

\[ T^{\mu_1 \cdots \mu_k} v_1 \cdots v_l \] are \( v_{\mu_1} \cdots v_{\mu_k} v_{\nu_1} \cdots v_{\nu_l} \).

As before, we can define (differentiable) tensor fields on a manifold. Their components (in a given basis) are then functions on the manifold

\[ T^{\mu_1 \cdots \mu_k} v_1 \cdots v_l (x^\alpha) \]
given a set of functions

\[ T^{\mu \nu \ldots \rho} v_1 \ldots v_k (x^0) \]  

and a coordinate system on the manifold, we can define a tensor

\[ T = T^{\mu \nu \ldots \rho} v_1 \ldots v_k \epsilon_{\mu \ldots \rho} \frac{\partial}{\partial x^\mu} \ldots \frac{\partial}{\partial x^\nu}. \]

**Exercise 9**

Show that the derivative of a vector

\[ \frac{\partial A^\mu}{\partial x^\nu} \]

does not transform like a tensor (under coordinate transformations).

In other words, we cannot use

\[ \frac{\partial A^\mu}{\partial x^\nu} = T^\mu \]

to define a tensor.

**Contraction**

Given a tensor \( T \) of rank \((k,l)\), we can define a new tensor by "contraction" of one covariant and one contravariant slot.
This gives a tensor of rank \((k-1, l-1)\).

In a coordinate basis:

\[
T^{m_1 \ldots m_k \ldots m_l} \rightarrow \tilde{T}^{m_1 \ldots m_{k-1} \ldots m_l} = T^{m_1 \ldots m_{k-1} \ldots m_l} \quad \text{(recall Einstein's summation convention)}
\]

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**Exercise 10**

Show that the components of a contracted "tensor" indeed transform like a tensor.

**Exercise 11** Show that

\[
T(v_1' \ldots v_k' \ldots v_l'; v_1 \ldots v_k) = \tilde{T}^{m_1 \ldots m_k \ldots m_l} v_1 \ldots v_l \rightarrow \tilde{T}^{m_1 \ldots m_{k-1} \ldots m_l} w_1 \ldots w_k \rightarrow \tilde{T}^{m_1 \ldots m_{k-1} \ldots m_l} v_1 \ldots v_k.
\]

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2.5 The metric

So far we have considered vectors, forms and tensors. We have not defined however any notion about the "length" of a vector.

**Definition**

A metric \(g\) on a manifold \(M\) is a symmetric, non-degenerate tensor field.
Symmetric:

\[ g(\nabla_1, \nabla_2) = g(\nabla_2, \nabla_1). \]

In components: \( g_{\mu\nu} = g_{\nu\mu} \).

Non-degenerate:

\( g(\nabla_1, \nabla_2) = 0 \quad \forall \nabla_2 \in T_p M \)

if and only if \( \nabla_1 = 0 \)

(The 0 vector maps any function to 0, and has all its components equal to zero).

In components:

\[ g_{\mu\nu} \nabla_1^\mu \nabla_2^\nu = 0 \quad \forall \nabla_2^\nu \]

if and only if \( \nabla_1^\mu = 0 \).

Often we write \( ds^2 \) instead of \( g \):

\[ ds^2 = g_{\mu\nu} dx^\mu dx^\nu \]

This is a generalization of Pythagoras in diff. form:

\[ dl^2 = dx^2 + dy^2 + dz^2. \]

The metric captures the notion of a scalar product on \( T_p M \) caution:
the metric is not necessarily positive definite.

Acting on a single vector, it defines the "length" of a vector

\[ ds^2 = g_{ij} dx^i (x) \, dx^j (x) \].

At any point of the manifold, we can always find a set of orthonormal vectors \( E_1, \ldots, E_n \) with

\[ g(E_i, E_j) = \delta_{ij} \]

The number of + and - signs does not depend on the particular orthonormal vectors we have chosen, and defines the signature of the metric:

\[ + \cdots + \] : Euclidean signature

\[ - + \cdots + \] : Lorentzian signature

\[ -- + \cdots + \] : Usually not considered.
Sporulina is endowed with a
metric of Lorentzian signature: \(-+++\).

Example

Minkowski space is the spacetime
of special relativity. Its metric is

\[ ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu, \quad \eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \]

The vectors \( e_\mu \) form an
orthonormal basis with

\[ ds^2 (e_\mu, e_\nu) = \eta_{\mu\nu}. \]

In general, we cannot find coordinates
in which the spacetime metric is that
of Minkowski. However, it is possible
always to choose coordinates \( x^\mu \) such
that, at a single point \( p \),
In such coordinate systems is locally flat. These coordinates correspond to those of a freely falling coordinate system, in which the effects of gravity have "disappeared" and spacetime appears to be that of special relativity.

Length of curves

The metric allows us to find the length of a curve \( \gamma(\lambda) \to X^\mu(\lambda) \).

Suppose the vector \( n_\lambda \) associated to the curve is "spacelike" at each point \( \gamma(\lambda) \):

\[ g(n_\lambda, n_\lambda) > 0. \]
Recall that the components of $\mathbf{u}$ are

$$\gamma^\mu = \frac{dx^\mu}{d\lambda} \Rightarrow \frac{dx^\mu}{d\lambda} = \frac{dx^\mu}{d\lambda}.$$  Hence

$$g(\gamma^\mu, \gamma^\mu) = g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} = \frac{ds^2}{d\lambda^2}.$$  

We have defined

$$S = \int_{\lambda_i}^{\lambda_f} ds = \int_{\lambda_i}^{\lambda_f} d\lambda \sqrt{g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}}.$$  

By assumption.

If the curve is "timelike",

$$g(\gamma^\mu, \gamma^\mu) < 0$$

we instead define the proper time along the curve to be

$$\tau = \int_{\lambda_i}^{\lambda_f} d\lambda \sqrt{-g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}}.$$
Exercise 12

Show that the length or proper time of a curve is parametrization-independent.

Exercise 13

The (Euclidean) metric on $S^2$ is

$$ds^2 = d\theta^2 + \sin^2 \theta \, d\phi^2.$$ 

i) "Derive" the form of this metric.

ii) Find an orthonormal basis of vectors (what is the metric signature?)

iii) Calculate the length of a circle

$$a + \theta = \theta_0 = \text{const} \quad (0 \leq \theta \leq \pi).$$