Last time

We introduced several concepts:

i) given a coordinate system $x$ on a manifold, we can represent functions on $M : f : M \rightarrow \mathbb{R}$ by functions acting at the corresponding coordinates:

$$f : M \rightarrow \mathbb{R}$$

$\uparrow$

$\tilde{f} : \mathbb{R}^n \rightarrow \mathbb{R}$

ii) given a coordinate system
We can also define a set of $n$ vectors $e_\mu$ by their action on the "coordinate representation" of a function:

$$e_\mu(f) = \frac{\partial f}{\partial x^\mu}.$$ 

(iii) The vectors $e_\mu$ form a basis of the vector space $T_p M$:

$$\nabla = \nabla^m e_\mu.$$ 

(iv) Under change of coordinate

$$\nabla' = \frac{\partial x'}{\partial x} \nabla.$$
v) Note that under a change of coordinates

\[
\frac{e_i'}{\partial x'} = \frac{\partial e_i}{\partial x'} = \frac{\partial x'}{\partial x'} = \frac{\partial x'}{\partial x'} = \frac{\partial x'}{\partial x'}
\]

Usually we do not work with "abstract" vectors, we work with the components of the vector \( v^m \) in a given coordinate basis.

We extend now the concept of a tangent vector \( v \) at a point \( p \) to a set of tangent vectors defined at each point of the manifold.

**Definition**

A (differentiable) **vector field** \( \mathfrak{v} \) is

a (differentiable) map \( \mathfrak{v}: p \in M \rightarrow \mathfrak{v}(p) \in T_p M \),
between points of the manifold and tangent vectors at those points.

Such a map is differentiable if for all differentiable functions \( f : M \rightarrow \mathbb{R} \), \( \nabla ( \cdot ) (f) \) is a differentiable function.

We can represent a (differentiable) vector field \( \nu \) by its components \( \nu^m(x^0) \), which now depend on the manifold coordinates. For a differentiable vector field, the components \( \nu^m(x^0) \) are differentiable.

Example

Suppose we are given a vector field \( \nabla \). At every point \( p \in M \), \( \nabla_p \) maps a function to \( \mathbb{R} \).
Therefore, \( \nabla (f) \) defines a function \( \nabla : \mathbb{R} \to \mathbb{R} \) from the manifold to \( \mathbb{R} \):

\[
\nabla (f) : \quad p \in M \to \nabla_p (f) \in \mathbb{R}.
\]

It hence makes sense to consider the commutator of two vector fields \( \nabla \) and \( \nabla \)

\[
[\nabla, \nabla] = \nabla \nabla - \nabla \nabla
\]

which defines a new vector field:

\[
[\nabla, \nabla] (f) = \nabla (\nabla (f)) - \nabla (\nabla (f))
\]

**Exercise 6**

1. Show that the commutator indeed defines a vector field (properties of a vector)
2. Show that in a coordinate basis,

\[
[\nabla, \nabla] = \nabla \otimes \nabla - \nabla \otimes \nabla
\]
2.3. Forms

The set of all vectors at \( p \in M \) defines the linear space \( T_pM \).

A dual vector (or one-form, or form) is an element of the dual space \( T^*_pM \). In other words:

A dual vector \( \omega \) is a linear map

\[
\omega : T_pM \rightarrow \mathbb{R}
\]

\[
\forall v \in T_pM \rightarrow \omega(v) \in \mathbb{R}
\]

The set of all \( n \) dual vectors at \( p \) defines another linear space, the cotangent space \( T^*_pM \).
Example

given a function on a manifold $f$, we can always define a one-form $df$ by

$$df(v) = v(f)$$

This form is analogous to the gradient of the function:

$df$ is a "function" that takes a vector $v$ and returns how fast $f$ is changing along $v$.

A coordinate system $\mathbf{x}$ is a set of $n$ functions on $M$:

$$x^\mu : M \to \mathbb{R} \quad (\text{fixed } \mu)$$

we can therefore define a set of
n 1-forms \( dx^\nu, \quad \nu = 1, \ldots, n \).

**Exercise 7**

i) Show (rigorously) that

\[ dx^\nu (\xi^\nu) = \delta^\nu_\nu \]

ii) The \( dx^\nu \) form a basis of \( T^*_p M \).

iii) By ii) we can write

\[ w = w^\nu dx^\nu. \]

Show that under coordinate transformations

\[ w^\nu' = \frac{\partial x^m}{\partial x'^\nu} w^\mu. \]

\[ dx'^\nu = \frac{\partial x^m}{\partial x'^\nu} dx^m. \]

The forms \( dx^\nu \) are elements of the dual coordinate basis.
Corollary

In a coordinate basis, the components of a vector \( \mathbf{v} \) are

\[ v^i = dx^i(\mathbf{v}) \]

Because of the way they transform,

- vectors \( \rightarrow \) contravariant vectors
- forms \( \rightarrow \) covariant vectors.

Just as before, we can define a form field to be a map

\[ \omega : M \rightarrow T^*_p M \]

\[ p \mapsto \omega(p) \in T^*_p M. \]
2.4. Tensors

We can construct additional linear spaces by tensor product:

Let $T_p^{(k, l)} M$ be the tensor product space

$$T_p^{(k, l)} M = (T_p^* M)^\otimes^k \otimes (T_p M)^\otimes^l,$$

or

$$T_p^{(k, l)} M = \{ \omega^1 \otimes \cdots \otimes \omega^k \otimes \nu^1 \otimes \cdots \otimes \nu^l \mid \omega^i \in T_p^* M, \nu^j \in T_p M \}$$

(the set of all combinations of $k$ forms and $l$ vectors).

Definition

A tensor of rank $(k, l)$ is a multilinear map

$$T : T_p^{(k, l)} M \to \mathbb{R}$$
\[ \omega \otimes \cdots \otimes \omega \otimes v_{1} \cdots \otimes v_{k} \rightarrow T(\omega', \ldots, \omega^k, v_1, \ldots, v_k) \in \mathbb{R} \]

"Multilinear" means linear in each component, with the remaining ones kept fixed:

\[
T(a \omega + \beta \omega', \omega^2, \ldots, \omega^k, v_1, \ldots, v_k) = \\
= a T(\omega, \omega^2, \ldots, \omega^k, v_1, \ldots, v_k) + \\
+ \beta T(\omega', \omega^2, \ldots, \omega^k, v_1, \ldots, v_k),
\]

and similarly for all other components.

**Examples**

- A form is a tensor of rank \((0,1)\).
- We can think of a vector as a tensor of rank \((1,0)\).
- A tensor of rank \((1,1)\) is a linear map from \(T_p M\) to \(T_p M\), or, alternatively, from \(T^*_p M\) to \(T^*_p M\).