Last time

we defined a vector (at a point) as a derivative operator. The vector maps functions defined in a neighborhood of \( p \) onto \( \mathbb{R} \), and has the properties of a derivative:

i) linearity  
ii) Leibniz rule

iii) \( \nabla (f) = \nabla (g) \) if \( f = g \) in neighborhood of \( p \).

Exercise 3

show that if \( f \) is constant in a neighborhood at \( p \), \( \nabla (f) = 0 \).
Example

Suppose we are given a curve \( \gamma : \mathbb{R} \to M \) that passes through a point \( p \in M \): \( \gamma(\lambda_p) = p \in M \)

We can construct a vector at the point \( p \), the vector "tangent" to the curve by the following prescription:

\[ \tau_{\gamma}(f) = \frac{d}{d\lambda} (f \circ \gamma) \bigg|_{\lambda_p} \]
Exercise 4

Show that $\gamma$ satisfies all the properties of a vector.

So far, our discussion has not relied on any coordinates.

Assume now that there are coordinates $x$ that cover the point $p$:

$$x: \mathbb{C} \rightarrow \mathbb{C} \times \mathbb{R}^n, \quad p \in \mathbb{C}.$$

- We can characterize the curve $\gamma$ by its coordinates $\dot{\gamma} = x \cdot \dot{x}$.
- We can also characterize any function $f$ by its coordinates:

$$f = f \circ x^{-1}$$
It follows that

\[ \nabla_V (f) = \left. \frac{d}{d\lambda} \left( f \circ \sigma \right) \right|_{\lambda^p} = \left. \frac{d}{d\lambda} \left( f \circ x^{-1} \circ x \circ \gamma \right) \right|_{\lambda^p} = \]

\[ = \left. \frac{d}{d\lambda} \left( \tilde{f} \circ \tilde{\gamma} \right) \right|_{\lambda^p} = \left. \frac{d}{d\lambda} \left( \tilde{f} \circ \tilde{\gamma} (\tilde{\gamma}(\lambda)) \right) \right|_{\lambda^p} = \]

\[ = \left. \frac{\partial \tilde{f}}{\partial x^m} \right|_{x^m = \tilde{\gamma}(\lambda)} \cdot \left. \frac{d\tilde{\gamma}^m}{d\lambda} \right|_{\lambda^p} \quad \text{(drop hides: sloppy notation)} \]

\[ = \left. \frac{df}{dx^m} \cdot \frac{dx^m}{d\lambda} \right|_{x^m(p)} \quad \text{(Recall Einsteins summation convention).} \]
Let us look at the object

\[
\frac{\partial f}{\partial x^\mu} \bigg|_{\mathbf{x}^\nu(p)} \quad (\text{actually, \textit{n} different objects, } \\
\mu = 1, \ldots, \textit{n})
\]

If we are given a point \( p \), with collective coordinates \( \{ x^\nu(p) \} \), we can define a \textit{n} curves through \( p \) by changing the \( \mu \)-th coordinate and keeping the remaining ones fixed:

\[
\gamma^\mu(\lambda) = x^{-1} \left( \{ x^\nu(p) + (\lambda - \lambda p) \delta^\mu{}^\nu \} \right)
\]

Clearly, \( \gamma^\mu(\lambda) \) goes through \( p \):

\[
\gamma^\mu(\lambda p) = x^{-1} \left( \{ x^\nu(p) \} \right) = p
\]
Exercise 5

Show that the vector $\mathbf{Y}$ associated with the curve $g^\alpha(x)$ (as defined on the example at page 2) acts on $f$ as

$$e_\mu(f) = \frac{\partial f}{\partial x^\mu} \bigg|_{x^\alpha(p)}$$

going back to the relation on page 4

$$\nabla_Y(f) = \frac{\partial f}{\partial x^\mu} \bigg|_{x^\alpha(p)} \frac{dx^\mu}{dx} \bigg|_{x^\alpha(p)} \cdot e_\mu(f) \frac{dx^\mu}{dx} \bigg|_{x^\alpha(p)}$$

arbitrary curve

we find that we have expressed the vector $\nabla_Y$ as a linear combination:

$$\nabla_Y = \frac{dx^\mu}{dx} \bigg|_{x^\alpha(p)} \cdot e_\mu$$
Definition

The tangent space of a manifold $M$ at a point $p$, $T_pM$, is the set of all tangent vectors at $p$.

Claim (see Wald for proof)

- $T_pM$ is an $n$-dimensional vector space ($n$ is also the dimension of the manifold)
- The vectors $\frac{\partial}{\partial x^i}$ are a basis of $T_pM$.

We speak of $\frac{\partial}{\partial x^i}$ as a coordinate basis of the tangent space, since these vectors are linked to a given coordinate chart. Because of their action on functions, sometimes people write

$$\frac{\partial}{\partial x^i} = \frac{\partial}{\partial x^i}.$$
Again: $e_\mu(f)$ tells us how fast the function $f$ changes along the $\mu$-th coordinate.

**Change of coordinates**

Because the $e_\mu$ form a basis at $T_pM$, we can express any vector $\vec{v} \in T_pM$ as a linear combination

$$\vec{v} = v^\mu e_\mu.$$

The $v^\mu$ are the components of the vector $\vec{v}$ in the coordinate basis $e_\mu$.

If the vector is tangent to a curve, we have for instance $v^\mu = \left. \frac{dx^\mu}{d\lambda} \right|_p$.

What happens to the components if we choose different coordinates $x'$?
The action of an arbitrary vector on a function \( f \) does not depend on any choice of coordinates, so

\[
\nabla (f) = \nabla^\nu \partial_\nu (f) = \nabla^\nu \frac{\partial f}{\partial x^\nu} = \nabla^\nu \frac{\partial f}{\partial x^\mu} \frac{\partial x^\mu}{\partial x^\nu} = \nabla^\nu \frac{\partial x^\mu}{\partial x^\nu} \nabla^\mu (f)
\]

Comparison then shows

\[
\nabla^\mu = \frac{\partial x^\mu}{\partial x^\nu} \nabla^\nu
\]

(vector transformation law)

(The derivative is evaluated at \( x^\nu(p) \).)

Typically, we do not work with the actual vector \( \nabla \), but with its coordinates (in an arbitrary coordinate basis) \( \nabla^\mu \).
The notion of tangent vector can be generalized to all points in the manifold:

**Definition**

A (differentiable) **vector field** \( \mathbf{v} \) is a **differentiable** map \( \mathbf{v} : p \in M \rightarrow \mathbf{v}(p) \in T_pM \) between points of the manifold and vectors at those points. Such a map is **differentiable** if for all **differentiable** functions \( f : M \rightarrow \mathbb{R} \),

\[
\mathbf{v}(\cdot)(f) \text{ is a differentiable function on } M
\]

We can represent a differentiable vector field by its components \( v^m(x^u) \), which now depend on the **coordinates** of the manifold points.