Black Holes

So far, we have restricted our attention mostly to $r > R_s = 2GM$, since we expect that for regular planets or stars $R_s < R_{\text{star}}$.

Today we shall study the Schwarzschild solution globally (for all $r$). As we shall see, the Schwarzschild solution is expected to describe the endpoint of the gravitational collapse of a sufficiently massive star.

Recall that the Schwarzschild metric

$$ds^2 = -(1 - \frac{2GM}{r})dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1}dr^2 + r^2d\Omega^2$$

has a coordinate singularity at $r = 2GM$. 
We can "expose" this coordinate singularity by changing to a new coordinate system in which the metric is regular at $r = 2GM$.

Then are many such coordinate systems.

Here we choose a system of coordinates which is regular everywhere, with the exception of the $R$ singularity at $r = 0$.

**Vinko Kal's extension**

Suppose that we introduce new coordinates implicitly defined by the relations

$$\left(\frac{r}{2GM} - 1\right) \exp\left(\frac{r}{2GM}\right) = R^2 - T^2 \quad (r)$$

$$\frac{t}{2GM} = 2 \arctanh\left(\frac{T}{R}\right) \quad (t)$$

($\theta$ and $\phi$ remain unaffected)
In these coordinates, the Schwarzschild solution becomes

\[ ds^2 = \frac{32 M^3}{r} \cdot e^{-\frac{r}{2GM}} \left( -dT^2 + dR^2 \right) + r^2 \, d\Omega^2 \]

with the understanding that \( r \) has to be expressed in terms of \( T \) and \( R \) using equation (r).

In this form, the metric has several useful features:

- **Radial**
  - i) Null geodesics obey \( dT = \pm dR \), as in Minkowski space.
ii) The metric is regular everywhere except at \( r = 0 \).

Note that \( r = 0 \) corresponds to the two "curves"

\[
T = \pm \sqrt{1 + R^2}.
\]

As we say, the curvature diverges at \( r = 0 \).

(iii) The Schwarzschild radius \( r = R_S = 2GM \) has coordinates \( R^2 = T^2 \Rightarrow R = \pm T \)

(iv) Lines of constant \( r \) have two forms:

a) \( r > 2GM \Rightarrow R = \pm \sqrt{c + T^2} \) (\( c \) a constant)

b) \( r < 2GM \Rightarrow T = \pm \sqrt{c' + T^2} \) (\( c' \) a constant)

(v) Lines of constant \( t \) satisfy

\[
\frac{T}{R} = \tanh \left( \frac{t}{4GM} \right) \Rightarrow T = \tanh \left( \frac{t}{4GM} \right) R
\]

constant slope with magnitude smaller than 1.
We thus arrive at the following spacetime diagram:

There are four regions, separated by the Schwarzschild radius $2\,M$:

the event horizon
I: Is the exterior of the black hole; at $r \to 0$ the space approaches Minkowski. In this region, an observer can avoid reaching the singularity at $r = 0$.

II: Is the interior of the black hole. No signal from region II can reach the exterior / region I. An observer in this region will unavoidably hit the singularity at $r = 0$. Note that $r$ is a timelike coordinate in II: The singularity at $r = 0$ is a spacelike singularity: it has spatial extent.

IV: Is a "parallel" universe; the exterior of the black hole "on the other side". There is no way to reach IV from I.

III: Is the rim over $\partial$ II. It is known as the white hole region. Any observer will finally escape this region, and there is no way for observers at I or IV to send signals to III.
How realistic is the Kinska solution?

Black holes should form by time evolution of appropriate initial conditions.

The initial conditions implicit in Kinska's extension, a white hole, are unphysical.

Instead, we expect black holes to emerge from the gravitational collapse of a sufficiently massive star.

Calculations of interior (spherically symmetric) equilibrium solutions of Einstein's equations show that there are limits on the total mass of a star that do not depend on the behavior of matter at high densities (the equation of state).
If the mass exceeds this limit, we expect the star to collapse into a black hole:

There is indirect evidence for the existence of black holes in our universe; compact objects that seem to exceed the mass limits on stable configurations.
To conclude, we should point out that in 1974 Hawking showed that
some to quantum-mechanical effects,
black holes redacte. Black holes are
not black, after all....

The End