Last time

After the development of special relativity, the "most natural" way to include gravitation would have been to look for a Lorentz invariant theory of gravity.

This was not the path followed by Einstein; mostly because of

1. The principle of equivalence

Let's go back to Newtonian mechanics:

2nd Newton's law:

\[ \vec{F} = m \vec{a} \quad , \quad m : \text{inertial mass} \]

Universal law of gravitation:

\[ \vec{F_g} = \frac{G m_1 m_2}{r^2} \quad , \quad m_1, m_2 : \text{masses} \]
(compare with electric force \( \vec{F}_E = q \cdot \vec{E} \))

The remarkable fact in Newtonian mechanics is that inertial mass and gravitational mass are the same:

\[ m_I = m_g \]

Therefore \( m \cdot \vec{a} = m \cdot \vec{g} \) \( \Rightarrow \) \( \vec{a} = \vec{g} \)

In a gravitational field all bodies fall with the same acceleration!

If we are confined to a freely-falling elevator, by studying the motion of objects around us we won't be able to tell that we are in a gravitational field!

Einstein elevated this observation to a principle:
1. It is impossible to detect the existence of a gravitational field by means of local experiments.

2. Locally, in a freely falling reference frame, the laws of nature are those of special relativity.

If there is no way to detect the existence of a "gravitational field" (under certain circumstances), it may be that there is actually no gravitational field.

Einstein's equivalence principle has a somewhat weaker formulation in the
Weak equivalence principle

The trajectory of a neutral test body in a given gravitational field only depends on its initial position and velocity, and is independent of its composition and internal structure.

The (weak) equivalence principle is one of the most well-tested principles in nature.

The experiments at the University of Washington show that the accelerations of two different test bodies (one made of Be, and the other made of Ti) satisfy

\[ \eta = \frac{21 \, \text{a Be} - \text{a Ti}}{1 \, \text{a Be} + \text{a Ti}} = (0.3 \pm 1.8) \times 10^{-13} \]

\( \Rightarrow \) Eötvös parameter
Since the trajectories of test bodies do not depend on their composition, these trajectories can be attributed to the geometry of spacetime.

In such a way, gravitation becomes a manifestation of geometry instead of a “regular” force.

2. The principle of general relativity (general covariance)

Usually we do not perform experiments in freely-falling labs. The link between the equivalence principle and the laws of nature in a non-freely-falling coordinate system is provided by a straightforward generalization of the principle of special relativity.
Principle of general relativity

The laws of nature have the same form in all coordinate systems.

The principle of general relativity (general covariance) plays a central role in modern physics. It is independent from the principle of equivalence: there are many generally covariant theories that do not respect the principle of equivalence (we'll see some of them later).

Einstein based General Relativity on both principles: general covariance, principle of equivalence. Hence, in order to prove to GR we need to know...
i) How to write down physical laws that take the same form in all coordinate systems.

ii) How to relate the motion of a body to the geometry of space-time.

Both i) and ii) lead us to differential geometry, the topic of the next few classes.

2. Differential Geometry

Our first task consists in coming up with the appropriate mathematical object to capture the notion of "space-time": A smooth four-dimensional space. Our general definition will also apply to many other important spaces.
2.1. Manifold

Basically, an n-dimensional manifold is a smooth space that locally looks like \( \mathbb{R}^n \).

\textbf{Definition} (see Carroll or Wald for more rigor)

An n-dimensional (real) differentiable manifold \( M \) is a set of points \( M \) together with a collection of subsets \( \{ O_\alpha \}_{\alpha=1}^3 \) such that

i) Each \( p \in M \) belongs at least to one \( O_\alpha \) (the \( O_\alpha \) cover \( M \)).
(ii) For each $\alpha$ there is a one-to-one map $x_\alpha : O_\alpha \subset M \rightarrow U_\alpha \subset \mathbb{R}^n$, where $U_\alpha$ is an open subset of $\mathbb{R}^n$. The map $x_\alpha$ is a coordinate system on the portion $\alpha$ of the manifold covered by $O_\alpha$.

(iii) If $O_\alpha \cap O_\beta$ is non-empty, the map $x_\beta \circ x_\alpha^{-1} : x_\alpha (O_\alpha \cap O_\beta) \cap U_\alpha \rightarrow x_\beta (O_\alpha \cap O_\beta) \cap U_\beta$ is differentiable.

(Not $\circ$ denotes function composition; $(x_\beta \circ x_\alpha^{-1})(p) = x_\beta (x_\alpha^{-1}(p))$. )
We shall assume that all subsets $O_{\alpha}$ and coordinate systems $x_{\alpha}$ compatible with coordinate i), ii) and iii) come within the manifold.

**Examples**

- $\mathbb{R}^n$ is a manifold

- The surface of a sphere (in 3d or arbitrary dimensions) is a manifold

$$S_2 = \{ \vec{x} \in \mathbb{R}^3 \mid \vec{x} \cdot \vec{x} = R^2 \}$$

$S_2$ cannot be covered by a single coordinate system.
- All Lie groups are manifolds
e.g. $\text{SO}(3)$, $\text{SU}(2)$, $\text{E}_8$, ...

- A closed interval $[a, b] \subset \mathbb{R}$ is not a manifold

- Two cones intersecting at their tips is not a manifold

- Orbifolds (manifolds whose with points under the action of a discrete group indented) are not manifolds (in general)