Last time

Gravity wave polarizations:

\[ e_R = \frac{1}{\sqrt{2}} (e_+ + i e_x) \quad ; \quad e_L = \frac{1}{\sqrt{2}} (e_+ - i e_x), \]

\[ e_+ = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad ; \quad e_x = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \]

for \( \vec{k} = (0, 0, k) \).

Under rotations around \( \tau \) axis (\( \parallel \vec{k} \))

\[
\begin{cases}
    e_R \to e^{2i\phi} e_R \\
    e_L \to e^{-2i\phi} e_L
\end{cases}
\]

Recall from QM: Spin (angular momentum) is the generator of rotations:

\[ R = e^{-i\vec{\omega} \cdot \vec{s}} \]
The projection of the spin of a gravity wave along the momentum vector $\mathbf{p}$ takes two values: $+2$, $-2$.

Gravitons have helicity $\pm 2$.

(Gravitons have spin 2).

Impact of gravitational waves on matter

In order to see how gravity waves affect matter it is going to be useful to derive the geodesic deviation equation:

Consider a one-parameter set of geodesics: $\gamma_s(t)$. For fixed $s$, $\gamma_s(t)$ is a geodesic with affine parameter $\lambda$.

This set of geodesics defines a two-dimensional (sub)manifold. We can choose $t$ and $s$
to be the coordinates on this manifold.

We can then define two vector fields on this two-dimensional submanifold:

\[ \frac{\partial}{\partial s} = \frac{\partial x^m}{\partial s} \quad \frac{\partial}{\partial \tau} = \frac{\partial x^m}{\partial \tau} \quad \epsilon^m \]

\[ \frac{\partial}{\partial s} = S^m \quad \frac{\partial}{\partial \tau} = T^m \quad \epsilon^m \]

\( T^m \) is tangent to the geodesics.

\( S^m \) is the deviation vector; it points to the "next" geodesic.
Define now the relative velocity between geodesics:
\[ V^\mu = (\nabla_\tau S)^\tau = T^\nu \nabla_\nu S^\mu \]

and the relative acceleration between geodesics:
\[ A^\mu = (\nabla_\tau V)^\tau = T^\nu \nabla_\nu V^\mu. \]

Note that the actual acceleration of a geodesic is zero:
\[ a^\mu = T^\nu \nabla_\nu T^\mu = 0 \quad \text{(geodesic equation)} \]

Since \( s, \tau \) are coordinates, \( \left[ \frac{\partial}{\partial s}, \frac{\partial}{\partial \tau} \right] = 0 \).

Using the definition of commutator (Exercise 6):
\[ S^\mu \nabla_\nu T^\nu - T^\nu \nabla_\nu S^\mu = 0 \]

and because torsion vanishes (with Exercise 22b)
\[ S^\mu \nabla_\nu T^\nu - T^\nu \nabla_\nu S^\mu = 0. \]
Exercise 40

By repeated use of the last equation, show that

\[ A^n = R^r_{upq} T^v T^p S^q. \]

Also,

\[ A^n = \frac{D^2 S^n}{dt^2} = R^r_{upq} T^v T^p S^q, \]

the geodesic deviation equation.

In the presence of curvature, initially parallel geodesics, \( \frac{DS^n}{dt} = 0 \), do not remain parallel.

Consider now the effect of a gravitational wave on such a set of neighboring geodesics:

\[ ds^2 = -dt^2 + \eta_{ij}(t,x) dx^i dx^j. \]
The Riemann tensor is at least linear in $h_{ij}$, so to leading order we can take

$$\mathbf{T} = (1, 0, 0, 0),$$

which implies $t = 2$.

The geodesic deviation equation hence becomes

$$\frac{\partial^2 \mathbf{s}}{\partial t^2} = R^{\mu}_{\nu \rho \sigma} \mathbf{s}^\nu.$$  Using

$$R_{\mu \rho \nu \sigma} = \frac{1}{2} (\mathbf{h}_{\rho \sigma}),$$

and because $h_{\rho \sigma}$ is spacelike,

$$\frac{\partial^2 \mathbf{s}}{\partial t^2} = \frac{1}{2} \mathbf{h}^{\mu \nu} \mathbf{s}_\nu.$$  Therefore, for a **polarized** wave,

$$h^{\mu \nu} = h_+ (e_+) \mathbf{1} e^{i \mathbf{k} \cdot \mathbf{x} - \omega t}, \quad e_+ = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\mathbf{S}^x = \frac{1}{2} S^x h_+ (e^{i \mathbf{k} \cdot \mathbf{x} - \omega t}), \quad \mathbf{S}^x = 0.$$  Similarly,

$$\mathbf{S}^\sigma = -\frac{1}{2} S^\sigma h_+ (e^{i \mathbf{k} \cdot \mathbf{x} - \omega t}).$$
The solution is

\[
\begin{align*}
S^x &= 1 + \frac{1}{2} \, h_+ \min (i \mathbf{k} \cdot \mathbf{x} - \kappa t) \, S_0^x(0) \\
S^2 &= 1 - \frac{1}{2} \, h_+ \min (i \mathbf{k} \cdot \mathbf{x} - \kappa t) \, S_0^2(0) \\
S^z &= S^z(0)
\end{align*}
\]

Note: · Motion along \( \tau \)-direction unaffected - gravitational waves are transverse · Symmetry under \( 180^\circ \) rotation - gravitational waves have spin 2

Similarly, for a \( x \)-polarized wave,

\[
E_x = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}
\]
\[
\begin{align*}
S_x &= S_x(0) + \frac{1}{2} \hbar x \min(\xi^2 - k^2) \ S_y(0) \\
S_y &= S_y(0) + \frac{1}{2} \hbar x \min(\xi^2 - k^2) \ S_x(0) \\
S_z &= S_z(0)
\end{align*}
\]

Exercise 4.1

Derive the equation of motion of a particle with deviation vector \( \mathbf{S} \) in the presence of a circularly polarized gravitational wave (right and left), and find the corresponding solution.

Note: The change in the displacement vector is
\[
\frac{\delta \mathbf{S}}{\mathbf{S}} \sim \hbar, \quad \text{the strain.}
\]
As you know, several groups are trying to detect gravitational waves. Most of these efforts involve gravitational wave interferometers.

\[ \text{Laser} \rightarrow \text{mirror} \]
\[ \downarrow \]
\[ \downarrow \]
\[ \text{mirror} \rightarrow \text{with phase difference} \]

**Exercise 42**

Consider two freely-falling test masses (an emitter and a mirror) under the influence of a gravitational wave.

\[ X_E(t) \rightarrow \quad X_M(t) \]
Assume that they are separated by 1 at time $t = 0$, and that they are initially at rest.

Calculate the phase shift in a light ray emitted at $t = 0$ which bounces off the mirror and comes back to the emitter to linear order in the gravitational wave amplitude.