Last time

Propagation of a photon in a weak gravitational field:

\[ \vec{x}(t) = \vec{x}(t) + \delta \vec{x}(t) \quad \text{with} \]

\[ \vec{x}(t) = \vec{x}_0 + \vec{\eta}(t - t_0) \]

\[ \delta \vec{x}(t) = \delta x_u(t) \vec{\eta} + \delta \vec{x}_\perp, \quad \delta \vec{x}_\perp \cdot \vec{\eta} = 0. \]

\[
\frac{d^2 \delta \vec{x}_\perp}{dt^2} = -(1 + \chi) \left[ \vec{\nabla} \phi - \vec{\eta} (\vec{\eta} \cdot \vec{\nabla} \phi) \right]
\]

\( \delta \vec{x}_\perp \) describes the deflection of the photon from its "straight" trajectory caused by the gravitational field.

Consider now the deflection caused by a massive object like a star:
Choosing boundary conditions such that
\[
\frac{d\delta \mathbf{x}_1}{dt} \bigg|_{t_m} = \mathbf{0}, \quad \text{since } \mathbf{\tilde{n}} \text{ has unit norm,}
\]

\[
\mathbf{x} = - \int_{t_m}^{t_{obs}} dt \frac{d^2 \delta \mathbf{x}_1}{dt^2} = (1+r) \int_{t_m}^{t_{obs}} dt \left[ \nabla \phi - \mathbf{\tilde{n}} (\nabla \mathbf{\cdot} \nabla \phi) \right].
\]

The gravitational potential due to the star is
\[
\phi (\mathbf{x}) = - \frac{6M}{|\mathbf{x}|} \quad \text{(star located at } \mathbf{x} = 0)\]

To leading order, we write the potential along the unperturbed trajectory: \( \mathbf{x} = \bar{x}_e + \mathbf{\tilde{n}} (t-t_e) \).

Since
\[
\frac{\partial \phi}{\partial x^i} = \frac{2}{\partial x^i} \left( - \frac{6M}{|\mathbf{x}|} \right) = \frac{2}{\partial x^i} \left( - \frac{6M}{(\bar{x}^2)^{1/2}} \right) = \frac{GM}{(\bar{x}^2)^{3/2}}
\]

\( (\nabla \phi)(\mathbf{x}(t)) = \frac{GM}{(\bar{x}(t))^{3/2}} \mathbf{x}(t) \) and hence
\[ [\nabla \phi - \vec{n} \cdot (\nabla \phi)] (\vec{x}(t)) = \frac{GM}{(\vec{x}(t))^3} \frac{(\vec{x}(t) - \vec{n} \cdot (\vec{n} \cdot \vec{x}(t)))}{b} \text{ (perpendicular component of trajectory)} \]

Hence

\[ \vec{a} = (1+\varepsilon) \int_{t_\text{os}}^{t_\text{fin}} dt \frac{GM}{(\vec{x}(t))^3} \vec{b} = (1+\varepsilon) \frac{GM}{b} \int_{t_\text{os}}^{t_\text{fin}} dt \frac{GM}{[(\vec{x}_\text{e} + \vec{n}(t-t_\text{e}))^2]^{3/2}} \]

\[ = (1+\varepsilon) \frac{GM}{b} \int_{t_\text{os}}^{t_\text{fin}} dt \frac{1}{(b^2 + (t-t_\text{e})^2)^{3/2}} \]

\[ = (1+\varepsilon) \frac{GM}{b} \left. \frac{t}{b^2 \sqrt{b^2 + t^2}} \right|_{t_\text{os} - t_\text{e}}^{t_\text{fin} - t_\text{e}} = \frac{2 (1+\varepsilon) GM b}{b^2} \]

\[ \alpha = \frac{2 (1+\varepsilon) GM}{b} \]

In GR \( \varepsilon = 1 \), so \( \alpha = \frac{4 GM}{b} \). The deflection of a light ray from a fixed star that just grazes the sun (\( b = R_\odot \)) was measured during
a solar eclipse by an expedition led by Eddington in 1919. The measurement agreed with the predictions of GR.

**Exercise 37**

Calculate the deflection angle (in arc sec) caused by the sun for $b = R_0$ (sun radius).

This is one of the three "classical tests" of GR.

There is an additional effect caused by the gravitational field: From the previous lecture:

\[ \vec{v} \cdot \frac{d\vec{x}}{dt} = (1 + \gamma) \Phi \quad \text{or} \quad \frac{d\Phi}{dt} = (1 + \gamma) \Phi \]

Since $\Phi < 0$, a light ray takes longer to cover a given distance: Shapiro's time delay
This is the effect exploited by the Cassini mission to place constraints on $\phi$.

To end our discussion of the scalar sector, consider the linearized equations in vacuum:

$$2 \nabla^2 \psi = 0 \quad \Rightarrow \quad \psi = 0$$

$$\partial_i \psi = 0$$

$$(\partial_i \partial^i - 2 \delta_{ij}) (\phi - \psi) + 2 \phi \delta_{ij} = 0 \quad \Rightarrow \quad \phi = 0.$$ 

In general relativity, the scalar sector is non-dynamical. (Metric perturbations are constrained to vanish).

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**Vector sector**

Consider now vector metric perturbations:

$$ds^2 = -dt^2 + \delta_{ij} \, dx^i \, dx^j + [\delta_{ij} + \partial_i \psi_j + \psi_i \psi_j] \, dx^i \, dx^j.$$ 

Under a gauge transformation with $\xi^\mu$,

$$h_{\mu \nu} \rightarrow h_{\mu \nu} - \partial_\mu \xi_\nu - \partial_\nu \xi_\mu.$$
$\epsilon^i$ is a scalar. We can decompose

$$\epsilon_i = \partial_i \epsilon + T \epsilon_i, \quad \partial_i T \epsilon_i = 0 \text{ (transverse)}$$

under gauge transformations with parameters $T \epsilon_i$,

$$\begin{cases}
\Delta S_i = -T \epsilon_i; \\
\Delta T_i = -T \epsilon_i;
\end{cases}$$

Thus, by choosing $T \epsilon_i = T_i$ we can set $T_i = 0$:

$$T_i \rightarrow T_i + \Delta T_i = T_i - T \epsilon_i = 0.$$ 

In this gauge the linearized Einstein's eqs are:

$$\begin{cases}
\delta G_{00} = 0 = \nabla^2 \Phi_0, \\
\delta G_{0i} = -\frac{1}{2} \nabla^2 S_i = 8 \pi G T_{0i}, \\
\delta G_{ij} = -\frac{1}{2} \nabla^2 S_{ij} - \frac{1}{2} \nabla^2 S_{ij} = 8 \pi G T_{ij},
\end{cases}$$

with

$$T_{ij} = (\rho + p) u_i u_j + pg_{ij},$$

$$= -p \delta_{ij} + O(\epsilon^2).$$
we find, with \( \mathbf{v}_i \), the transverse component of \( \nu_i \)

\[
\nabla^2 S_{ij} = 16\pi G \rho \nu_{ij}, \quad \text{or}
\]

\[
S_{ij}^{(x)} = -4G \int d^3x' \frac{\rho \nu_{ij}^{(x')}}{|\mathbf{x} - \mathbf{x}'|}
\]

To linear order in the velocity of the test particle, the geodesic equation is

\[
\frac{d^2 x^i}{dt^2} + \Gamma^i_{00} + 2\Gamma^i_{0j} \nu^j = 0.
\]

In the vector vector, \( \Gamma^i_{00} = S_i \)

\[
\Gamma^i_{0j} = \frac{1}{2} \left( \partial_j S^i - \partial^i S_j \right).
\]

Hence

\[
\frac{d^2 x^i}{dt^2} = - S_i - \left( \partial_j S^i - \partial^i S_j \right) \nu^j.
\]

Define now \( \vec{E} = - \nabla S \) and

\[
\vec{B} = \nabla \times \vec{S} \quad (B_i = \epsilon_i^{jk} \partial_j S_k)
\]
Exercise 28

Show that the geodesic equation can be cast as

$$\frac{d^2 \vec{x}}{dt^2} = \vec{E} + \vec{\nabla} \times \vec{B}$$

"Gravitomagnusm". The geodesic equation takes the form of the Lorentz force, with

$$\vec{F} = m (\vec{E} + \vec{\nabla} \times \vec{B}).$$

The vector $\vec{S}$ plays the role of the vector potential $\vec{A}$. Adding the contribution of the scalar potential $\phi$ we have the geodesic equation

$$\frac{d^2 \vec{x}}{dt^2} = \vec{E} + \vec{\nabla} \times \vec{B},$$

with

$$\vec{E} = -\vec{\nabla} \phi - \vec{S}, \quad \vec{B} = \vec{\nabla} \times \vec{S} \quad \text{exactly as in EM}!$$
Consider now an observer that parallel-transport a spatial vector $\mathbf{V}$:

$$ u^m \nabla_m V^i = 0, \quad u^m V_m = 0 \quad \text{(spatial)} $$

Since to lowest approximation $u^i = (1, 0)$ we can write

$$ u^0 \nabla_j V^i = \dot{V}^i + \Pi^i_{\ j} V^j = 0 $$

or, as before

$$ [\Pi^i_{\ j} = \frac{1}{2} \left( \partial_j s^i - \partial^i s_j \right)] $$

$$ \frac{\partial \mathbf{V}}{\partial t} = \frac{1}{2} \mathbf{V} \times \mathbf{B} $$

where $\mathbf{B} = \nabla \times \mathbf{J}$, $\mathbf{B}(\mathbf{x}) = -4 \pi \int d^3 x' \ \frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|}$

Therefore, the spatial vector $\mathbf{V}$ precesses, with angular velocity $\frac{\mathbf{B}}{2}$. This is known as the Lense-Thirring effect, also known as frame dragging.
Consider a massive planet like the Earth which rotates around its axis
with angular velocity \( \omega \). Then, on dimensional grounds and because of
symmetry,

\[
\vec{B} = \frac{GM R \hat{w}}{R R} = \frac{GM}{R} \hat{w}.
\]

Thus, the rotation of the planet drags along inertial frames.

One of the main goals of gravity probe B is to detect this effect.

To conclude: In vacuum \( \vec{S} = 0 \)

\( \Rightarrow \) The vector vector of general relativity

is non-dynamical.