Last Time

Non-relativistic weak field limit of \( \phi \):

\[
d s^2 = - (1 + 2\phi) \, dt^2 + (1 - 2\phi) \, dx^2.
\]

\[
A = 4\pi \phi, \quad \psi = \phi; \quad \frac{d^2 x_i}{d t^2} = - \frac{2\phi}{2\xi}.
\]

Post-Newtonian Limit and PPN Formalism

There is a systematic way to analyze the non-relativistic limit of any theory of gravity that respects the equivalence principle:

Consider the action for a point-particle:

\[
S = -m \int dt \sqrt{-g_{\mu\nu} \frac{dx^\mu}{d t} \frac{dx^\nu}{d t}} = -m \int dt \sqrt{g_{\mu\nu} \mathbf{v}^\mu \mathbf{v}^\nu},
\]

where \( \mathbf{v}^\mu = \frac{dx^\mu}{d t} \) is the speed.

\( g_{\mu\nu} = -g_0 - 2g_0^i \xi \mathbf{v}^i \mathbf{v}^k \)
We uncover the equations of Newtonian mechanics if we set

\[ S = -m \int dt \left( 1 + 2\phi \frac{\dot{r}^2}{r^2} \right)^{1/2} \]

We then assign to \( \dot{r}^2 \) and \( \phi \) the leading order \( \epsilon \) in a small-velocity expansion: \( \phi = O(\epsilon) \), \( \dot{r}^2 = O(\epsilon) \).

Indeed, for a particle in a circular orbit,

\[ \dot{r}^2 = \frac{6M}{r} = -\phi. \]

Newtonian mechanics includes terms of \( O(\epsilon) \) in the action. To go beyond Newtonian mechanics we need the action up to terms of \( O(\epsilon^2) \) [terms of \( O(\epsilon^{1/2}) \) and \( O(\epsilon^{3/2}) \) are forbidden by symmetry].

\[ L = 1 + 2\phi - \dot{r}^2 - g_{00} [O(\epsilon^2)] - 2g_{\mu\nu} [O(\epsilon^{3/2})] \dot{r}^\mu \dot{r}^\nu - g_{\mu\nu} [O(\epsilon)] \dot{r}^\mu \dot{r}^\nu. \]
Example: Determine $g_{ij}$ to $O(\epsilon)$. 

Recall the linearized eqs.:

$$2 \Delta^2 \Psi = 8\pi G \left[ (\rho + p)u^0 u_0 + -p \right] + \ldots$$

$$\partial_j \Psi = 8\pi G \left[ (\rho + p)u^0 u_j \right] + \ldots$$

$$(\delta_{ij} \Delta^2 - \partial_i \partial_j)(\phi - \Psi) + 2 \delta_{ij} \ddot{\Psi} = 8\pi G \left[ (\rho + p)u_i u_j + p \delta_{ij} \right] + \ldots$$

We know from the Newtonian limit that $\rho = O(\epsilon)(= O(\phi))$. To leading order

$$u_0 = -\frac{dt}{d\tau} = -\frac{1}{\sqrt{1 - \dot{v}^2}}$$

with $\dot{v} = \frac{dx}{dt}$

$$= 1 + O(\epsilon)$$

Because pressure arises from relativistic motion, $p = O(\epsilon^2)$. Hence, to $O(\epsilon)$ we find

\[
\begin{cases}
2 \Delta^2 \Psi = 8\pi G (\rho + O(\epsilon^2)) \\
\partial_j \Psi = 0 + O(\epsilon^{3/2}) \\
(\delta_{ij} \Delta^2 - \partial_i \partial_j)(\phi - \Psi) + 2 \delta_{ij} \ddot{\Psi} = 0 + O(\epsilon^2)
\end{cases}
\]
Since \( \frac{\partial}{\partial t} = \nabla \cdot \mathbf{v} \), \( \frac{\partial}{\partial t} = O(\epsilon^{1/2}) \).

Therefore, to \( O(\epsilon) \) we find

\[
\psi = \phi, \quad \Delta \phi = 4\pi G \rho \quad \text{as before.}
\]

In particular in \( 6R \) \( \psi = 8\phi \) with \( \delta = 1 \).

The parameter \( \gamma \) is one out of ten Post-Newtonian parameters that can be used to describe the gravitational field of a matter distribution. In any theory of gravity that respects the equivalence principle, in Brans-Dicke,

\[
\gamma = \frac{1+w}{2+w}.
\]

See attached table from C.M. Will,

The Confrontation between general Relativity and Experiment, Living Reviews in Relativity.
Box 2. The Parametrized Post-Newtonian formalism

1. **Coordinate system**: The framework uses a nearly globally Lorentz coordinate system in which the coordinates are \((t, x^1, x^2, x^3)\). Three-dimensional, Euclidean vector notation is used throughout. All coordinate arbitrariness ("gauge freedom") has been removed by specialisation of the coordinates to the standard PPN gauge (TEGP 4.2 [147]). Units are chosen so that \(G = c = 1\), where \(G\) is the physically measured Newtonian constant far from the solar system.

2. **Matter variables**: 
   - \(\rho\): density of rest mass as measured in a local freely falling frame momentarily comoving with the gravitating matter;
   - \(v^i = (dx^i/dt)\): coordinate velocity of the matter;
   - \(w^i\): coordinate velocity of the PPN coordinate system relative to the mean rest-frame of the universe;
   - \(p\): pressure as measured in a local freely falling frame momentarily comoving with the matter;
   - \(\Pi\): internal energy per unit rest mass. It includes all forms of non-rest-mass, non-gravitational energy, e.g. energy of compression and thermal energy.

3. **PPN parameters**: \(\gamma, \beta, \xi, \alpha_1, \alpha_2, \alpha_3, \zeta_1, \zeta_2, \zeta_3, \zeta_4\).

4. **Metric**: 
   \[
   g_{00} = -1 + 2U - 2\beta U^2 \quad - 2\xi \Phi_W + (2\gamma + 2 + \alpha_3 + \zeta_1 - 2\xi)\Phi_1 \\
   + 2((3\gamma - 2\beta + 1 + \xi)\Phi_2 + (2(1 + \zeta_1)\Phi_3 + (3\gamma + 3\alpha_1 - 2\xi)\Phi_4 \\
   - (\zeta_1 - 2\xi)\Lambda - (\alpha_1 - 2\alpha_2 - \alpha_3)w^2U - \alpha_2w^2U_{ij} + (2\alpha_3 - \alpha_1)w^2V_i \\
   + O(\epsilon^2)),
   \]
   \[
   g_{0i} = -\frac{1}{2}(4\eta + 3 + \alpha_1 - 2\alpha_2 + \zeta_1 - 2\xi)V_i - \frac{1}{2}(1 + \alpha_2 - \zeta_1 + 2\xi)W_i \\
   - \frac{1}{2}(\alpha_1 - 2\alpha_2)w^2U - \alpha_2w^2U_{ij} + O(\epsilon^2),
   \]
   \[
   g_{ij} = (1 + 2\gamma U + O(\epsilon^2))\delta_{ij}.
   \]

5. **Metric potentials**: 
   \[
   U = \int \rho \frac{d^3x'}{|x - x'|^3},
   \]
   \[
   U_{ij} = \int \rho \frac{(x - x')(x - x')}{|x - x'|^5} d^3x',
   \]

---

31. The Confrontation between General Relativity and Experiment

\[
\Phi_W = \int \rho \frac{d^3x'}{|x - x'|^3} \left( \frac{x' - x''}{|x - x'|^3} - \frac{x - x''}{|x' - x'|^3} \right) d^3x'd^3x'',
\]
\[
\Phi_1 = \int \rho \frac{d^3x'}{|x - x'|^3} d^3x',
\]
\[
\Phi_2 = \int \rho \frac{d^3x'}{|x - x'|} d^3x',
\]
\[
\Phi_3 = \int \rho \frac{d^3x'}{|x - x'|} d^3x',
\]
\[
\Phi_4 = \int \rho \frac{d^3x'}{|x - x'|} d^3x',
\]
\[
V_i = \int \rho \frac{d^3x'}{|x - x'|} d^3x',
\]
\[
W_i = \int \rho \frac{d^3x'}{|x - x'|} (x' - x') d^3x'.
\]

6. **Stress-energy tensor** (perfect fluid):

\[
T^{00} = \rho(1 + \Pi + \epsilon + 2U),
\]
\[
T^{0i} = \rho v^i(1 + \Pi + \epsilon + 2U + p/\rho),
\]
\[
T^{ij} = \rho v^i v^j(1 + \Pi + \epsilon + 2U + p/\rho) + \rho \delta^{ij}(1 - 2\gamma U).
\]

7. **Equations of motion**: 
   - Stressed matter: \(T^{\mu\nu} = 0\),
   - Test bodies: \(d^2x^\mu/d\lambda^2 + \Gamma^\mu_{\nu\lambda}(dx^\nu/d\lambda)(dx^\lambda/d\lambda) = 0\),
   - Maxwell's equations: \(F^{\mu\nu} = 4\pi J^\mu\), \(F_{\mu\nu} = A_{\nu\mu} - A_{\mu\nu}\).
The parameter $\gamma$ is important because it affects the propagation of photons:

For a null geodesic:

\[
\begin{cases}
\frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu_{\rho\sigma} \frac{dx^\rho}{d\lambda} \frac{dx^\sigma}{d\lambda} = 0 \quad \text{and} \\
g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} = 0
\end{cases}
\]

**Exercise 3.7**

Show that the geodesic equation can be written

\[
\frac{d^2x^i}{dt^2} + \left( \Gamma^i_{\mu\nu} - \Gamma^\mu_{\nu i} \frac{dx^\mu}{dt} \frac{dx^\nu}{dt} \right) \frac{dx^\mu}{dt} \frac{dx^\nu}{dt} = 0
\]

using $\Gamma^\mu_{\nu i} = \frac{\partial \phi}{\partial x^i}$

$\Gamma^i_{\mu j} = -\gamma \phi \delta_{ij}$, $\Gamma^i_{\mu 0} = \frac{\partial \phi}{\partial x^i}$

$\Gamma^i_{0 j} = -\gamma \phi \delta_{ij}$, $\Gamma^i_{j 0} = -\gamma \left( \frac{\partial \phi}{\partial x^i} \delta^i_j + \frac{\partial \phi}{\partial x^0} \delta^i_j - \frac{\partial \phi}{\partial x^j} \delta^i_0 \right)$

$\gamma = \frac{1}{\sqrt{g^{00}}}$
Exercise 38

Show that to leading order in the post-Neutonian expansion

\[ \frac{d^2 \vec{x}}{dt^2} = - \vec{\nabla} \phi \left( 1 + \gamma \left( \frac{d\vec{x}}{dt} \right)^2 \right) + 2 \frac{d\vec{x}}{dt} \left( \frac{d\vec{x}}{dt} \cdot \vec{\nabla} \phi \right) (1 + \gamma) \]

\[ 0 = 1 + 2 \phi - \left( \frac{d\vec{x}}{dt} \right)^2 (1 - 2 \gamma \phi) \]

To zeroth order a solution of these eqs. is

\[ \vec{x}(t) = \vec{x}_e + \vec{n}(t - t_e) \quad \text{(straight line)}, \]

with

\[ \vec{n} \cdot \vec{n} = 1 \quad \text{(speed of light)}. \]

Writing \( \vec{x}(t) = \vec{\delta x}(t) \) and substituting

\[ \frac{d^2 \vec{\delta x}}{dt^2} = - \vec{\nabla} \phi \left( 1 + \gamma \right) + 2 \vec{n} \left( \vec{n} \cdot \vec{\nabla} \phi \right) (1 + \gamma) \]

\[ = -(1 + \gamma) \left[ \vec{\nabla} \phi - 2 \vec{n} (\vec{n} \cdot \vec{\nabla} \phi) \right] \quad \text{and} \]

\[ 0 = 1 + 2 \phi - (\vec{n}^2 + 2\vec{n} \frac{d\vec{x}}{dt}) (1 - 2 \gamma \phi) \]
or

\[ \hat{n} \frac{d \delta \vec{x}}{dt} = (1 + \gamma) \phi. \]

The photon trajectory is thus

\[ \vec{x}(t) = \vec{x}(t) + \delta \vec{x}(t) = \vec{x}_e + \hat{n}(t-t_e) + \delta \vec{x}(t) \]

and we impose \( \delta \vec{x}(t_e) = 0 \), so that

\[ \vec{x}(t_e) = \vec{x}_e. \]

Decompose this trajectory into a component parallel to \( \hat{n} \), \( \delta \vec{x} = \hat{n} \cdot \vec{x}(t) \) and perpendicular to \( \hat{n} \), \( \delta \vec{x}_\perp = \vec{x} - \hat{n} (\hat{n} \cdot \vec{x}) \).

By construction, \( \hat{n} \cdot \delta \vec{x}_\perp = 0 \), and

\[ \delta \vec{x} = \delta \vec{x}_\parallel \hat{n} + \delta \vec{x}_\perp \]

Substituting this expansion into the propagation equations

\[ \frac{d^2 \delta x_\parallel}{dt^2} \hat{n} + \frac{d^2 \delta x_\perp}{dt^2} = -(1 + \gamma) \left[ \hat{\nabla} \phi - 2 \hat{n} (\hat{n} \cdot \hat{\nabla} \phi) \right] \]
\[ = - (1 + \gamma) \left[ -\hat{n} \left( \hat{n} \cdot \nabla \phi \right) + \nabla \phi - \hat{n} (\hat{n} \cdot \nabla \phi) \right] \]

Parallel to \( \hat{n} \)

Perpendicular to \( \hat{n} \)

Hence

\[ \frac{d^2 \hat{x}}{dt^2} = - (1 + \gamma) \left[ \nabla \phi - \hat{n} (\hat{n} \cdot \nabla \phi) \right]. \]

Consider the deflection caused by a massive body like a star, \( \phi = - \frac{GM}{r} \).

Using the unperturbed trajectory of the photon,

\[ \phi = - \frac{GM}{|\hat{x}_e + \hat{n}(t-t_e)|} = - \frac{6M}{|\hat{x}(t)|} \]

(the star is at \( \hat{x} = 0 \)). The gradient is

\[ \frac{\partial \phi}{\partial x_i} = \frac{\partial}{\partial x_i} \left( - \frac{6M}{|\hat{x}|} \right) = \frac{\partial}{\partial x_i} \left( - \frac{6M}{(\hat{x})^3} \right) = GM \frac{x_i}{|\hat{x}|^3} \]

\[ \nabla \phi \bigg|_{\hat{x}(t)} = \frac{GM}{|\hat{x}(t)|^3} \cdot \hat{x}(t) \quad \text{and} \]

\[ \frac{d^2 \hat{x}}{dt^2} = - (1 + \gamma) \left[ \frac{GM}{|\hat{x}(t)|^3} \cdot \hat{x}(t) \right]. \]
\[
\left[ \vec{\nabla} \phi - \vec{\eta} (\vec{\eta} \cdot \vec{\nabla} \phi) \right]_{X(t)} = \frac{GM}{|\vec{x}(t)|^{3/2}} \left( \vec{x} - \vec{\eta} \cdot (\vec{\eta} \cdot \vec{x}) \right)
\]

Consider the deflection caused by the star:

\[
\vec{n} + \frac{d\delta \vec{x}_N}{dt} \bigg|_{X=0} = \vec{n}_s + \frac{d\delta \vec{x}_S}{dt} \bigg|_{X=0}
\]

\[
\vec{b} = \vec{x} - \vec{n} \cdot (\vec{\eta} \cdot \vec{x})
\]

\[
\text{Impact parameter}
\]

\[
\sin \theta = \frac{d\delta \vec{x}_N}{dt} \bigg|_{X=0} \quad \text{deflection angle}
\]

\[
\hat{d} = \frac{d\delta \vec{x}_N}{dt} \bigg|_{X=0} - \frac{d\delta \vec{x}_S}{dt} \bigg|_{X=0} = (1 + \gamma) \int_{\text{en}}^\text{ass} \frac{GM}{|\vec{x}(t)|^{3/2}} \vec{b} =
\]

\[
= (1 + \gamma) \int_{\text{en}}^\text{ass} \frac{1}{\left( (\vec{x}_e + \vec{n} (t-t_e))^2 \right)^{3/2}} =
\]

\[
= (1 + \gamma) \int_{\text{en}}^\text{ass} \frac{1}{(b^2 + t^2)^{3/2}} = (1 + \gamma) \int_{\text{en}}^\text{ass} \frac{t}{b^3 \sqrt{b^2 t^2 + t_e^2}}
\]
So finally

\[ \alpha = \frac{Z(1+\delta)GM}{b} \]

In general relativity, \( \delta = 1 \), so \( \alpha = \frac{4GM}{b} \).

This prediction of the \( 6\sigma \) (made by Einstein) was verified during a solar eclipse by a team led by Eddington in 1919. It is one of the three "classical" tests of general relativity.