So far:

Spacetime is described by a Riemannian manifold:

A manifold endowed with a metric-compatible, torsion-free connection.

What determines the metric?

3. General Relativity

Our goal is to write down a set of equations that determine the spacetime metric, and the evolution of any other field (e.g. electromagnetism) in any given spacetime. These equations should satisfy:

1. General Covariance:
The equations should have the same form in all coordinate systems.
2. Equivalence principle:

The effects of "gravity" should disappear in a freely falling coordinate system.

3. Action principle:

We should be able to quantize the theory. The equations should follow from extremizing an appropriate action (the integral of a Lagrangian).

In order to satisfy 3, we need an action functional \( S = S [g_{\mu\nu}, \phi] \)

\( g_{\mu\nu} \): metric; \( \phi \): "matter fields" (e.g., photons, electrons, ...)

Property 1) demands that the action be a scalar under general coordinate transformations.
Property 2:) demands that the metric be the only field responsible for "gravitational interactions". Matter should couple "minimally" to this metric: in the matter action at special relativity, upscale \( \eta_{\mu \nu} \to g_{\mu \nu} \)

\[ S_{m}[\eta_{\mu \nu}, \Psi] \to S_{m}[g_{\mu \nu}, \Psi] \]

(if this is the case, the laws of physics take their special relativity form in a freely falling frame.)

We can classify the scalars constructed out of the metric according to the number of derivatives they contain:

0 derivatives - \( \Lambda \), a constant
At low energies (for slowly varying fields), the terms with the least number of derivatives give the largest contributions. This singles out the Ricci scalar $\mathcal{R}$.

Recalling our definition of invariant volume element, we find that the dynamics of gravity should be described by the action

$$S_g = \int d^4 x \sqrt{-g} \left[ \frac{\mathcal{R}}{16\pi G} + \Lambda \right]$$

$G$ is Newton's constant, $[G] = [M]^{-2}$

$\Lambda$ is cosmological constant.
By construction, $S_g$ is a scalar under coordinate transformations.

To construct the matter action, simply couple it "minimally" to $g_{\mu\nu}$:

$$S_m = \int d^4x \sqrt{-g} \; L_m (g_{\mu\nu}, \psi)$$


**Example:**

**in special relativity**

The Lagrangian of $EM^\psi$ is

$$L = -\frac{1}{4} \; \eta^\mu \eta^\nu + \frac{1}{4} \; \eta^\mu \eta^\nu$$

$$\mathcal{F}_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

Therefore,

$$S_{EM} = \int d^4x \; \sqrt{-g} \; \left(-\frac{1}{4}\right) \; g^\mu\rho \; g^\nu\sigma \; \mathcal{F}_{\mu\nu} \cdot \mathcal{F}_{\rho\sigma}$$

with

$$\mathcal{F}_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$
Exercise 25

Show that
\[ F_{\mu\nu} = \partial_{\mu} A_\nu - \partial_{\nu} A_\mu = \nabla_{\mu} A_\nu - \nabla_{\nu} A_\mu \]

Is this true in the presence of fermions?

Show that the action of electromagnetism is invariant under conformal transformations:
\[ g_{\mu\nu} \rightarrow \Lambda^2(x) g_{\mu\nu} \]

Overall, the action of spacetime electrocity coupled to matter is

\[ S = \int d^4x \sqrt{-g} \left[ \frac{R}{16\pi G} + \Lambda \right] + \int d^4x \sqrt{-g} L_m[g_{\mu\nu}, \Phi] \]

\( \Lambda \) (the cosmological constant) was introduced and then removed by Einstein. Cosmological observations may suggest \( \Lambda \approx 10^{-29} \text{ g/cm}^3 \). We shall set \( \Lambda = 0 \) for simplicity.
Exercice 26

Show that variation of the action above leads to Einstein's equations:

\[ G_{\mu\nu} = 8\pi G T_{\mu\nu}, \]

where

\[ T_{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta S_m}{\delta g_{\mu\nu}}. \]

\( T_{\mu\nu} \) is known as the energy-momentum tensor. Einstein's equations are consistent only if \( \nabla^\mu T_{\mu\nu} = 0 \) (the EMT is covariantly conserved).

Exercice 27

Show that \( \nabla^\mu T_{\mu\nu} \) follows from invariance of \( S_m \) under general coordinate transformations.
The energy-momentum tensor describes the distribution of energy and momentum of matter. The action for a collection of (non-interacting) particles is

$$S_m = \sum_i m_i \int d\lambda \sqrt{-g_{\mu\nu}} \frac{dx^i}{d\lambda} \frac{dx^i}{d\lambda}$$

$\text{mass of } i\text{-th particle}$

**Exercise 28**

Show that the corresponding EMT is

$$T^\mu_\nu(x) = \frac{1}{\sqrt{-\det g}} \sum_i \frac{p_i^\mu p_i^\nu}{m_i} \delta^{(4)}(x - x_i(\tau)) d\tau$$

$$= \frac{1}{\sqrt{-\det g}} \sum_i \frac{p_i^\mu p_i^\nu}{p_i^0} \delta^{(3)}(\vec{x} - \vec{x}_i(t))$$
Thus, in an inertial coordinate system:

\[ T^{00} = \sum_i p^i = \delta^{00}(x - \bar{x}; (t)) = \]

\[ = \sum_i \delta^{(3)}(\bar{x} - \bar{x}; (t)) \]

- \( T^{00} \) is energy density.

\[ T^{0i} = \sum_i p^i \delta^{(3)}(\bar{x} - \bar{x}; (t)) \]

- \( T^{0i} \) is momentum density in \( j \) direction.

\[ T^{kj} = \sum_i \frac{p^{k(i)} p^j(i)}{p^0(i)} \delta^{(3)}(\bar{x} - \bar{x}; (t)) = \sum_i \frac{dx^j}{dt} \frac{dx^k}{dt} \delta^{(3)}(\bar{x} - \bar{x}; (t)) \]

- \( T^{kj} \) is flow of momentum along the \( k \) direction across the surface \( x^j = \text{const} \).
In an arbitrary coordinate system:

- \( T_{\mu} u^\mu \) is energy density measured by an observer with four-velocity \( u^\mu \).

- If the vector \( e^\mu \) is orthonormal to \( u^\mu \), \( u^\mu e^\nu \equiv 0 \), \( e^\mu \) points along a spatial direction for the observer. Then

- \( T_{\mu} e^\mu \) is momentum density along direction \( e^\mu \).

- \( T_{\mu} e^\mu e^\nu \) is momentum flow along direction \( e^\mu \) across direction \( e^\nu \).