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Syracuse: 1949–1952

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9.1 Introduction

The year 1949 saw the publication of two papers by Peter Bergmann: *Non-Linear Field Theories* in the Physical Review [1] and, together with Johanna Brunnings, in the Reviews of Modern Physics [2], with the ambitious title *Non-Linear Field Theories II: Canonical Equations and Quantization*. These papers lay the foundation for the research of the Syracuse group working to quantize Einstein's theory of general relativity. A year later saw the publication by Paul Dirac of two papers in the Canadian Journal of Mathematics [3,4] which were based on a series of lectures he gave in Vancouver in August and September of 1948. These papers, while not concerned with general relativity, dealt with the problem of constructing a canonical formalism for a theory with constraints among the momenta and configuration space variables. Later that year, Pirani and Schild published *On the Quantization of Einstein's Gravitational Field Equations* [5]. This was their construction of the Hamiltonian for general relativity based on the ideas put forward by Dirac. This amazing flurry of work was all independent except for the stimulation by Dirac of Pirani and Schild. Dirac's papers do not mention general relativity. According to Felix Pirani [6], he and Alfred Schild attended theVancouver lectures. Alfred immediately saw the connection with general relativity and suggested that Felix work with him on the problem for his doctoral dissertation.

Although the work in Syracuse began without knowledge of the Dirac lectures or that Pirani and Schild existed, I feel that it is important for me to comment and to compare the different basic ideas which led to different constructions of the Hamiltonian. Dirac's first work in general relativity was published in 1958 [7,8]. It is very closely related to ideas in the Vancouver lectures, but very different from the other constructions. I will take the lecturer's prerogative of including material which is outside my defined limits because the ideas belong inside. Therefore, I shall begin with a comparison of Bergmann and Dirac approaches with emphasis on what theoretical ideas motivated them. Then I will discuss the work at Syracuse in the time frame of my title.

9.2 Fundamental Motivation

Peter Bergmann began his research with the intent of bringing together Einstein's theory of gravitation with the quantum theory of fields. General relativity is a non-linear field theory in which the underlying space-time geometry is not specified. The field equations, which determine that space-time geometry, are covariant with respect to continuous coordinate transformations which are piecewise four times differentiable. Quantum field theory, as we know it, is defined on a flat given space-time background, generally Minkowski space. In this background, one can construct a Hamiltonian and the associated Poisson brackets which can be carried over to the commutation relations of the operators of the quantum field theory. For general relativity, the field is the metric. It most certainly is not flat. The first problem for a quantum theory of gravity, then, is whether one can construct a Hamiltonian without having a background space-time. Then, having done so, can one find the Poisson bracket structure for observables and the appropriate Hilbert space that can lead to a generalized quantum field theory for the gravitational field. This is the task which Peter set for himself upon arriving at Syracuse University in 1947.

In the 1949 paper he states:

> The purpose of the present program is to analyze each of the two theories for its essential and, presumably relatively permanent contributions to our present knowledge and, thus, to construct what might be called skeletonized theories. An attempt will be made to see whether such a covariant theory is at all susceptible to quantization and whether the result will be an improved theory.

However, his particular approach to the problem came from his work as an assistant to Einstein from 1936 to 1940. Therefore, he goes on to say,

> ··· the theory of relativity contains two great permanent achievements: (a) it is the only theory of gravitation which explains reasonably the equality of inertial and gravitational mass (the so-called principle of equivalence); (b) it is the only classical field theory in which the equations of motion of particles in the field are contained in the field equations, instead of being logical juxtapositions.

The latter statement was based on the then recent determination of the equations of motion for compact sources by Einstein, Infeld, and Hoffmann [9–11]. Therefore, it was important that one treat the full non-linear theory. In a linearized, perturbative, version with a background Minkowski space, the essential character of general relativity is lost. First of all, with linearization, a flat rigid structure is substituted for the relation between matter and geometry, the crucial property of general relativity. Second, with linearization one loses, not only the self interaction, but also the interaction between source and field which leads to limitations on the motion of matter. While one may build up these interactions by successive approximation, the important structure of the unperturbed field is buried in the formalism of approximation. Furthermore, strong fields, where quantum interactions should be very important, are not small deviations from Minkowski space.

Therefore, Peter wished to study a general class of non-linear field theories which are covariant under the group of general coordinate transformations, the invariance group, and whose field equations define the motion of particles. Covariance refers to the existence of a group of transformations which leave the field equations unchanged, but which depend on a number of arbitrary functions of the space-time coordinates. As a result, a solution of the field equations can be mapped by an invariant transformation to another solution which remains the same on an initial surface, but is different in the future. Thus, the propagation of initial data is not unique. This property results in certain identities among the field equations themselves as will become evident.

The two goals, of covariance and equations of motion for particles, led him to study a general field theory whose field equations are derived from the variation of an action whose Lagrangian density is a function of generalized variables $y_A(x)$, $A = 1 \cdots N$ and their first derivatives. It is understood that the metric tensor describing the underlying geometry is included among the y_A. Arbitrary variations of the field variables in the action,

$$S = \int_V L(y_A, y_{A,\rho}) d^4x, \tag{9.1}$$

which vanish on the boundary of the four-dimensional domain, V, lead to the field equations

$$\begin{aligned} L^A &\equiv \partial^A L - (\partial^{A\rho} L)_{,\rho} = 0, \\ \partial^A L &\equiv \frac{\partial L}{\partial y_A} \qquad \partial^{A\rho} L \equiv \frac{\partial L}{\partial y_{A,\rho}}. \end{aligned} \tag{9.2}$$

He assumed that the Lagrangian is covariant and the field equations invariant under the variation induced by a general coordinate transformation that he writes as

$$\bar{\delta} y_A = u_{A\mu}{}^{\nu} \xi^{\mu}_{\nu} - y_{A,\mu} \xi^{\mu}; \tag{9.3}$$

the Lie derivative for the transformation $\delta x^{\mu} = \xi^{\mu}$; $u_{A\mu}{}^{\nu}$ is linear in the field variables, in general with constant coefficients. These transformations can be shown to form a group and, because they depend on arbitrary functions, the descriptors $\xi^{\mu}(x)$, they lead, as noted above, to differential identities, strong conservation laws, and to constraints among the momenta and field variables. The identities and constraints are a reflection of the differentiability of the mappings induced by the transformations.

Further, the EIH result shows that the field equations determine the motion of the sources of the field. As a result, before one has a solution, one cannot predict where the particle sources of the field might be located. Therefore, Peter introduced a parametric description of the coordinates, in terms of which he wished to describe the motion,

$$x^{\mu} = x^{\mu}(t, u^i), \qquad i = 1 \cdots 3. \tag{9.4}$$

It was Peter's hope that quantization of the Einstein theory would lead to the Schrödinger equation or it's generalization for particles. The introduction of parameters led to difficulties which were overcome in two long papers [2, 12]. However, the parameters produce four additional constraints, including the Hamiltonian. As a result, they did not lead to additional degrees of freedom for particles and the idea was quickly dropped.

Rather than the concern with non-linear field theories and general covariance, Dirac's motivation came from Lorentz invariant theories with constraints. In reference [3], he set a particle-like model in an N-dimensional configuration space with the assumption that the velocities can not be solved for in terms of the coordinates and the momenta,

$$p_n = \frac{\partial L(q, \dot{q})}{\partial \dot{q}^n}, \quad n = 1 \cdots N.$$

As a result there are constraints

$$\phi_m(q, p) = 0, \quad m = 1 \cdots M < N, \tag{9.5}$$

among the coordinates and momenta. Dirac distinguishes between *strong* and *weak* equations with zero right-hand sides. The variation of a strong equation with respect to its variables remains zero whereas the variation of a weak equation does not. The product of two weak equations is a strong equation, but there are other possibilities as well. In Dirac's usage, equations involving coordinates, velocities, and momenta may be strong.

As a result of the existence of constraints, he finds that the Hamiltonian is not unique, but one can always add a linear combination of the constraints with coefficients which may depend on the velocities as well as the coordinates and momenta. Propagation of these constraints leads to further constraints $\chi_k = 0, k = 1 \cdots K$. The totality of constraints can then be divided into first class constraints, whose Poisson brackets with all the constraints vanish modulo the constraints, and second class constraints whose Poisson brackets with other constraints do not vanish modulo the constraints. Only the first class constraints contribute to the Hamiltonian and the second class constraints can be eliminated as redundant canonical degrees of freedom. Dirac's motivation is in understanding the algebraic structure of the constraints, not in the existence of a group of invariant transformations. However, he, too, introduces an arbitrary parameter so that the time can become dynamical. He shows that this leads to the Hamiltonian as a constraint so that no new degree of freedom is added to the theory.

In the second paper, Dirac introduces a field theory in Minkowski space. He is not thinking about general covariance. But, he is concerned with Hamiltonian theories with constraints and with maintaining the four-dimensional symmetry of Lorentz invariance, while at the same time introducing an arbitrary space-like surface in terms of which to define the canonical formalism. To accomplish this, he introduces a parametric description of the Minkowski space coordinates as in (19.4) above. In order to assure that the surface $t = 0$ is space-like, Dirac introduces the time-like unit normal l_μ with the properties

$$l_\mu \frac{\partial x^\mu}{\partial u^i} = 0, \qquad l^\mu l_\mu = 1.$$

Then he proceeds to define quantities on the surface which are covariant under a change in parameters, which leaves the surface $t = 0$ unchanged. For tensors, these are the normal components to the hypersurface and the forms in the hypersurface. For example,

$$V_L = V^{\mu} l_{\mu} \quad \text{and} \quad V_{\mu} \frac{\partial x^{\mu}}{\partial u^{i}} = V_i . \tag{9.6}$$

Clearly, the decomposition of the field with respect to the time-like normal vector of the arbitrary space-like surface reflects the Lorentz invariance of the theory. This geometric decomposition of variables led to his particular treatment of the metric tensor when he later came to discuss general relativity. Peter referred to this decomposition as "D-invariance."

I do not want to discuss this further as it will take me too far from my purpose in discussing the work of the Syracuse group. But, I think it is important to see the difference between Peter Bergmann's view and goal and that of Paul Dirac at this time. Peter was thinking about quantization of a non-linear field theory which is covariant under a group of arbitrary coordinate mappings, in which the metric is part of the dynamical stucture, and in which the equations of motion for particles will be determined. Dirac is interested in theories in Minkowski space that may have constraints either imposed or intrinsic. Dirac is thinking more algebraically and Bergmann more group theoretically. Both introduce a parametric description of the coordinates, but ultimately conclude that it is useless.

It was Alfred Schild who saw that, once one introduced an arbitrary space-like surface, the Dirac formalism could be applied to general relativity. In a straightforward application of the Dirac approach with some clever mathematical manipulations, together with Felix Pirani he constructed the first explicit expression for the Hamiltonian [5]. However, they did not complete the decomposition of the metric or of the field variables for the Maxwell field with respect to l^{μ}, the normal. More important, they did not examine the propagation of the constraints until later [13].

In the following sections, I will sketch the results of the Bergmann group in the early period, 1949–52. First I discuss the derivation of conservation laws and equations of motion from the invariance under diffeomorphisms. Then, in Section 4, the construction of the Hamiltonian in the parameter formalism will be presented. In Section 5, the parameters will be dropped and the secondary constraints will be examined. Section 6 will examine how the results obtained in the canonical formalism appear in the Lagrangian formalism. Finally, in Section 7, preliminary steps to a quantum theory of gravity will be described.

9.3 Invariance, Conservation Laws, and Equations of Motion

If the field equations are to be invariant in form under a mapping $y_A(x) \rightarrow y_A(x) + \bar{\delta} y_A(x)$, the Lagrange density should be unchanged in form except for the addition of a total divergence. Thus, in general we have

$$\bar{\delta} L = L^A \bar{\delta} y_A(x) + (\partial^{A\rho} L \bar{\delta} y_A(x))_{,\rho} = Q^\rho{}_{,\rho}.$$

This can be rewritten as

$$L^A \bar{\delta} y_A(x) = (Q^\rho - \partial^{A\rho} L \bar{\delta} y_A(x))_{,\rho}. \tag{9.7}$$

Thus, when the field equations are satisfied, invariant transformations give us a conservation law. If the mapping is defined by (19.3), after integrating over an arbitrary four dimensional domain with a descriptor ξ^μ which vanishes on the boundary, we find an identity for the field equations and, as as a result, a *strong* conservation law:

$$\begin{aligned} (u_{A\mu}{}^{\nu} L^A)_{,\nu} + L^A y_{A,\mu} &\equiv 0, && \text{(9.8a)} \\ T^\nu{}_{,\nu} &\equiv 0, && \text{(9.8b)} \\ T^\nu = Q^\nu - \partial^{A\nu} L \, \bar{\delta} y_A &- u_{A\mu}{}^{\nu} L^A \xi^\mu. \end{aligned}$$

The identity in (19.8b) implies the existence of a superpotential such that

$$T^\nu = U^{[\nu\mu]}{}_{,\mu}, \tag{9.9}$$

the square brackets imply skew symmetry. From the definition of T^ν, we get an expression for the field equations in terms of the superpotential:

$$\begin{aligned} u_{A\mu}{}^{\nu} L^A \xi^\mu &= -U^{[\nu\mu]}{}_{,\mu} + t^\nu, \\ t^\nu &= \partial^{A\nu} L \bar{\delta} y_A - Q^\nu. \end{aligned} \tag{9.10}$$

Note that for $\nu = 0$, the right-hand side is free of second time derivatives. Thus, this combination of the field equations corresponds to the secondary constraints in the canonical formalism.

If there are sources P^A which are not described by the fields, so that $L^A = -P^A$, the above equation becomes a relation between the sources and the field:

$$u_{\mu A}{}^{\nu} P^A \xi^\mu = U^{[\nu\sigma]}{}_{,\sigma} - t^\nu. \tag{9.11}$$

Thus, the strong conservation law becomes a conservation law for matter interacting with the field:

$$(u_{\mu A}{}^{\nu} P^A \xi^\mu + t^\nu)_\nu = 0.$$

Assume that there are several localized sources — even point particles — and consider a space-like surface in which a two-surface surrounds one of the local sources. In (19.10), let $\nu = 0$, integrate the resulting expression over the space-like interior of the two-surface, and obtain

$$\oint_{\partial V} U^{[0s]} n_s dS = \int_V (u_{\mu A}{}^{0} P^A \xi^\mu + t^0) dV.$$

This relates the matter and field in the interior to a surface integral which involves only field variables. Next in (19.10), let $\nu = s$ and integrate the resulting expression over the two surfaces to obtain

$$\oint_{\partial V} U^{[s0]}{}_{,0} n_s dS = \oint_{\partial V} (u_{\mu A}{}^{s} P^{A} \xi^{\mu} + t^{s}) n_s dS.$$

For localized sources P^A vanishes on the surface, so the expression says that the rate of change of some parameters defining the matter variables are related to the flux of matter through the surface. Through the field equations, one can show that the conditions on the sources are independent of the surface as long as the matter is confined within the surface. The relations above depend on the choice of descriptor ξ^{μ}. We may loosely think that choosing a vector normal to the space-like surface leads to an energy condition, while vectors on the surface lead to conditions on momentum or angular momentum.

These are the equations of motion for matter in general relativity [14]. As applied by Einstein, Infeld, and Hoffmann in the slow motion approximation, this leads to the equations of motion for point particles. This is also the basis for the calculations by Damour and Deruelle [15], and Iyer and Will [16].

9.4 The Parameter Formalism

The Bergmann group worked with the Einstein Lagrangian which is homogeneous quadratic in the first derivatives of the metric. Therefore, in the more general discussion, the Lagrangian was assumed to have the form

$$L = \Lambda^{A\rho B\sigma} y_{A,\rho} y_{B,\sigma},$$

and when parameters are introduced (dot= $\partial/\partial t$, $|s = \partial/\partial u^s$),

$$L' = JL, \qquad J \equiv \det\Big(x^{\mu}{}_{|s}, \dot{x}^{\mu}\Big) \tag{9.12}$$

which is clearly homogenous of degree 1 in the derivatives with respect to t.

With respect to the mappings (19.2), the Lagrangian is assumed to be of the same functional form except for the addition of a divergence. The invariance of the Lagrangian leads to identities among the field equations. These identities show that the definition of the momenta, $\pi^A = \partial JL/\partial \dot{y}_A$, cannot be inverted to solve for $\dot{y}_A$. As a result there are constraint equations involving only $y_A and \pi^A$ which I shall write as

$$\phi_{\mu}(\pi^A, y_A, y_{A|s}, x^{\rho}{}_{|s}) = 0. \tag{9.13}$$

In addition, invariance under parameter changes yield four more constraints for the momenta conjugate to the coordinates, $\lambda_{\rho} = \partial JL/\partial \dot{x}^{\rho}$,

$$\lambda_{\rho} - Jt_{,\rho}L + y_{A,\rho}\pi^A = 0.$$

These latter equations plus the homogeneity of the Lagrangian with respect to the velocities tell us that the Hamiltonian is zero:

$$H = \lambda_{\rho}\dot{x}^{\rho} + \pi^A \dot{y}_A - JL = 0,$$

which, as usual, one can show is independent of the velocities. Thus, there exists a function $H(\pi^A, y_A, y_{A|s}, \lambda_\rho, x^\rho, x^\rho{}_{|s}) = 0$ which generates the field equations. However, this Hamiltonian is not unique for one can add to it a linear combination of all the constraints.

This lack of uniqueness arises because the relationship between the momenta and velocities is singular. Therefore, one cannot invert the relation defining the momenta. However, a quasi-inverse can be found. The quasi-inverse also is not unique and it has null vectors which can be paired with those for the matrix connecting the velocities to the momenta. This lack of uniqueness can be exhibited explicitly and the Hamiltonian was constructed [12]. Thus, the lack of uniqueness results in the addition of an arbitrary linear combination of the constraints to the Hamiltonian. Up to this point, the only constraints are the *primary* constraints, those which come directly from the definition of the momenta.

Once one had the Hamiltonian, one formally formed the commutation relations based on the Poisson brackets and, looking forward to the quantum theory, said that the quantum state vector must vanish when operated on by the Hamiltonian or any of the constraints. Of course, such a formal statement was not quantization, but only an indication of the direction future work to quantize the field should go.

This is all I want to say about the theory with parameters. One soon found that the parameters were an unnecessary complication and after the fall of 1950 they were abandoned. As a result, the issue of *secondary* constraints was not examined in the parameter formalism.

9.5 Secondary Constraints

The construction of the Hamiltonian without the use of parameters was carried out by Robert Penfield [17]. He followed the technical arguments of the parameter construction. It was much simpler because one did not have the coordinates as additional variables and, therefore, no coordinate constraints. It also meant that the Hamiltonian did not have to vanish although the ambiguity with respect to the primary constraints remained.

Penfield also studied the propagation of the primary constraints [18]. He found that the propagation led to additional constraints. He recognized that these secondary constraints were just those field equations which lacked second time derivatives and therefore could be expressed directly in terms of the π^A and y_A. Propagation of the secondary constraints was carried out in several different ways [14,19,20] with the result that there were no tertiary constraints with the invariance group as defined above.

In discussing the secondary constraints, the question arose as to how they are related to the number of time derivatives in the transformation group. Therefore, Anderson and Bergmann [19] studied a generalized transformation group defined by

$$\bar{\delta} y_A = \sum_{p=0}^{P} f_{Ai}{}^{\mu_1\cdots\mu_p} \xi^i_{,\mu_1\cdots\mu_p} - y_{A,\rho}\delta x^\rho. \tag{9.14}$$

Here the index $i = 1 \cdots I$ ranges over the number of descriptors which includes the coordinate transformations, but may be more general. The main object here was to determine the relationship between the number of derivatives of the descriptors, particularly time derivatives, and the number of constraints which result in the canonical formalism. First of all, assuming the Lagrangian only contains first time derivatives, one obtains again the condition that the relation between momenta and time derivatives cannot be inverted. There exist I null vectors for the matrix relationship and, as a result, I primary constraints. It follows that although the momenta involves $\dot{y}_A$, their transformation under (19.14) contains only P time derivatives of the descriptors. Therefore, while the generator of a canonical transformation may contain $P+1$ spatial derivatives, it will contain only P time derivatives of the descriptors. Since the generator of the canonical transformation is an integral over the spacelike surface $x^0 =$ constant,

$$C = \int \mathcal{C} d^3x,$$

all the spatial derivatives of the descriptors ξ^i can be removed by an integration by parts, so that one can write (the index (p) indicates the number of time derivatives of ξ^i)

$$\mathcal{C} = \sum_{i=1}^{I} \sum_{p=0}^{P} {}^{p}A_i \xi^{ip}.$$

Assuming that (19.14) is an invariant transfomation group, it follows that the change in the functional form of the Hamiltonian is a linear combination of the primary constraints. That change is also given by the total time derivative of the generating functional which now contains time derivatives of the descriptors up to the $P + 1$th order. That is,

$$\delta' H = \int \delta w^i g_i = \frac{\partial C}{\partial t} + \left[C, H \right]$$

where the $g_i = 0$ are the primary constraints. The change in $\dot{y}_A$ as a result of the transformation is the same as that which results from the change in the Hamiltonian. One concludes that δw^i contains a term linear in the $P + 1$th time derivative of ξ^i. Therefore, one finds that ${}^{P}A_i = 0$ are the primary constraints and the remaining ${}^{p}A_i = 0$ are the secondary constraints which may also contain a combination of primary constraints. So there are $P + 1$ levels of constraints with I constraints at each level. That means that for the coordinate transformations, where $P = 1$ and $I = 4$, there are two levels and eight constraints in total. Unfortunately, this argument does not give us an explicit answer for the Poisson brackets between constraints for at each step the answer is arbitrary up to a linear combination of the primary constraints.

9.6 Lagrangian Formalism

Around this time, Feynman [21] and Schwinger [22] had begun to formulate quantum field theory through the action using the configuration and velocity field functions as the dynamical variables. Therefore, Bergmann and Schiller undertook to investigate how the results obtained in the canonical formalism appear in the Lagrangian formalism. They undertook to study more general transformations than the coordinate transformations. Therefore, they asked for the changes in the Lagrangian due to transformations which could involve point transformations on configuration space or on the velocity space as well as the coordinate transformations. Thus, they considered a transformation $\bar{\delta} y_A = f_A$ where f_A may depend on first derivatives of the field variables as well as the y_A. These are not necessarily invariant transformations, but, if solutions are mapped into solutions, the Lagrangian may change by the addition of a divergence as well as in form. Thus, they arrive at (19.7). By restricting the transformation so that the functional form of the Lagrangian does not change by the appearance of second time derivatives in the Lagrangian, one arrives at conditions on the f_A, part of which read

$$\partial^{A\cdot\cdot} Q^0 - \partial^{A\cdot} f_B \pi^B = 0, \pi^B = \partial^{A\cdot\cdot} L, \qquad \partial^{A\cdot\cdot} \equiv \frac{\partial}{\partial \dot{y}_A}. \tag{9.15}$$

Here, π^A is understood to be a function of $y_A, \cdots, y_A$. This condition restricts the transformations to the canonical transformations of the Hamiltonian formalism. If the definition of π^A can be inverted to eliminate $\dot{y}_A$, the generating density can be defined so that

$$f_A = \partial_A(\pi^B f_B - Q^0), \qquad \delta_A = \frac{\partial}{\partial \pi^A}. \tag{9.16}$$

With appropriate normalization of f_A and Q^ρ relative to a quasi-inverse [20], the above equation remains valid even in the singular case.

The above considerations are true whether or not the transfomation leaves the functional form of the Lagrangian unchanged. If one requires covariance for the theory, the results of Section 3 are recovered. When one expands the identity (19.8a), one finds that the term with third time derivatives will vanish only if $u_{A\mu} = u_{A\mu}{}^0$ satisfies

$$u_{A\mu} \Lambda^{AB} = 0, \qquad \Lambda^{AB} \equiv \partial^{A\cdot} \partial^{B\cdot} L. \tag{9.17}$$

Now, if we ask for the change in π^A as a function of $\dot{y}_A$ we find

$$\delta \pi^A = \Lambda^A B \delta \dot{y}_B, \tag{9.18}$$

so that $\delta \pi^A = 0$ for $\delta \dot{y}_A = \lambda^\mu u_{A\mu}$. This exhibits the extent of the indeterminacy of $\dot{y}_A$ for given π^A. Equation (19.18) also tells us that the momenta satisfy four primary constraints. (In general, the number of primary constraints is equal to the number of independent arbitrary functions in the invariance group.) Note, though, that if the definition of the momenta is substituted, these constraints vanish identically.

We recognize that the generating density defined in (19.16) is the zero component of (19.7). Thus, the generating function is the zero component of a differential conservation relation when the field equations are satisfied. The conserved quantity is the generating functional (Σ is the surface $x^0 =$ constant):

$$C = \int_{\Sigma} (Q^0 - \partial^{A0} L \bar{\delta} y_A(x)) d^3x. \tag{9.19}$$

In general, Q^{ρ} can be written

$$Q^{\rho} = Q_{\mu}{}^{\rho} \xi^{\mu} + Q_{\mu}{}^{\rho\sigma} \xi^{m}{}_{\sigma}. \tag{9.20}$$

A term with the second derivative of ξ has been omitted for simplicity. It's inclusion exhibits the lack of uniqueness of the resulting superpotential through the addition of the divergence of a quantity antisymmetric in three indices. The argument is easily extended to include this term, but it complicates the description without fixing uniqueness.

Substitute (19.20) into (19.10) with $U^{[\rho\sigma]} = U_{\mu}{}^{[\rho\sigma]} \xi^{\mu}$ to obtain

$$\begin{aligned} u_{A\mu}{}^{\rho} L^A &= -U_{\mu}{}^{[\rho\sigma]}, \sigma + t_{\mu}{}^{\rho}, & (9.21a) \\ U_{\mu}{}^{[\rho\sigma]} &= Q_{\mu}{}^{\rho\sigma} - \partial^{A\rho} L u_{A\mu}{}^{\sigma}, & (9.21b) \\ t_{\mu}{}^{\rho} &= Q_{\mu}{}^{\rho} + \partial^{A\rho} L y_{A,\mu}. & (9.21c) \end{aligned}$$

With these identifications we find that the generating functional can be written

$$\begin{aligned} C &= \textstyle\int_{\Sigma} \{-U_{\mu}{}^{[0\sigma]} \xi^{\mu}, \sigma - (Q_{\mu}{}^{0} + \partial^{A0} L y_{A,\mu}) \xi^{\mu})\} d^3x, \\ &= \textstyle\int_{\Sigma} \{-U_{\mu}{}^{[00]} \dot{\xi} + (U_{\mu}{}^{[0s]}{}_{,s} + t_{\mu}{}^{0}) \xi^{\mu} d^3x. \end{aligned} \tag{9.22}$$

Note that although $U_{\mu}{}^{[00]}$ is identically zero when expressed in terms of $y_A, \cdots y_A$, when the momenta, π^A are introduced, these become the primary constraints. The remaining terms are those field equations lacking second time derivatives, hence the secondary constraints. Thus the generating functional is a linear combination of the primary and secondary constraints. With appropriate boundary conditions, C is a constant of the motion. In that case, when one takes the time derivative of C, no additional constraints arise. This argument can be generalized as with Anderson and Bergmann. Again one finds that the number of primary and secondary constraints is the number of descriptors times the number of time derivatives plus one which appear in the invariant transformation law.

It is interesting that in all these arguments one starts from the fact that the Hamiltonian itself is not zero and one finds that the Hamiltonian is arbitrary up to a linear combination of the primary constraints. However, the existence of secondary constraints in the generating functional expresses the fact that the Hamiltonian also involves the secondary constraints with arbitrary coefficients. However, if the Lagrangian differs from a scalar density by a divergence, as in the case of general relativity, then the resulting Hamiltonian differs from zero also by a divergence.

In general relativity, the Hamiltonian density found is related to $t_{\mu}{}^{0}$ and that expression differs from the secondary constraints by a divergence of the superpotential (see (19.21a) with $\rho = 0$).

9.7 Quantization

In the transition to quantum theory, Peter wanted to keep as close as possible to the procedure in standard quantum field theory while recognizing that the ingredient of a fixed background spacetime would be missing. The formalism, however, depends on the existence of an invariant transfomation group. This transformation group is generated by constraints on the phase space. Therefore, the first objective was to try to preserve the algebra of the constraints in terms of the dynamical variables as operators on a Hilbert space rather than as functions or functionals on a phase space. Thus, the constraints are to keep their role as generators of invariant transformations. So, one begins by assuming that the basic Poisson brackets for the variables y_A and π^A go over to the commutation relations

$$[y_A(\mathbf{x}),\ \pi^{\mathbf{A}}(\mathbf{x}')] = \mathbf{i}\hbar\delta^{\mathbf{3}}(\mathbf{x} - \mathbf{x}'),$$

where $\mathbf{x}$ is a point on the hypersurface $x^0 =$ constant. Then one hoped to find a factor ordering of the constraints such that the algebra of constraints found by Bergmann and Anderson could be carried over to the quantum algebra. That proved to be difficult and later Jim Anderson proved that such a factor ordering did not exist if the constraints were put in a formally Hermitian order [23]. On the other hand, Artie Komar has argued that the constraints should not become Hermitian operators [24]. Nonetheless, the constraints, how ever they may carry over as operators, are the generators of the invariant transformations and as such their algebra should be carried over to the quantum theory if at all possible. In the Ashtekar formalism, this problem appears to be resolved [25].

Observable quantities must be invariant under the transformation group. In classical theory, we can establish a frame for observers and give a particular solution in a chosen frame. We know how to construct measurable quantities, the observables, relative to a given frame with a given solution. In a quantum theory of gravity, the dynamical variables become operators and are no longer attached to a particular solution or frame, but are representative of all solutions consistent with a particular Hilbert space. That is, there is no prior frame to which to attach the operators. Therefore, the quantum operators themselves must be constructed out of invariants. So *observables* were defined to be those functionals whose commutators with the constraints vanish modulo the constraints themselves. The search for observables began in the classical theory where the algebra was known and the problem of factor ordering is not a problem. With this definition, the constraints and the Hamiltonian are observables, the constraints being trivial zero observables. But while the Hamiltonian differs from a constraint only by a divergence, the non-zero Hamiltonian is also an observable and should be related to global energy. Dirac also recognized this fact in his definition of H_{main} [26]. However, no other observables were found. But, we know that in gravity there are four independent degrees of freedom per space point apart from the constraints and the Hamiltonian. There should be observables associated with these degrees of freedom.

As observables, the constraints and the Hamiltonian were assumed to become Hermitian operators. However, the notion of Hermiticity depends on the definition of the

Hilbert space and the measure defined on that space. There are problems even if one could find a space of functionals with a measure so that a norm is defined. Not only do the basic variables $(y_A,\ \pi^A)$ have c-number commutation relations, but so do the constraints and their canonical conjugates. Such quantities presumably have continuous spectra. But the constraints are to yield only zero when acting on a physically meaningful state vector. So one solution was that the Hilbert space itself should consist only of functionals for which the constraints are trivial operators. That is, all functionals in the Hilbert space, are annihilated by the constraints. Quantities conjugate to the constraints do not act on this space.

All of the above remarks are valid equally for canonical quantization and the Lagrangian procedure discussed by Bergmann and Schiller [20]. But, there are differences. The Lagrangian is constructed out of operators and the variations of the operators are also operators. First of all, the variations cannot be moved freely to the right side of each expression. Secondly, the operator Lagrangian is already a two-index object. If the variations were fully arbitrary, the operator field equations and conservation laws would become four -index objects. Therefore, they restricted the variations to the invariant transformations which depend on c-number descriptors. In this way they were able to recover the principal results of the canonical formalism. In addition, they were able to show how to derive the appropriate commutation relations for the dynamical variables and their time derivatives. However, at that time it was not clear whether the approach from the Lagrangian would lead to a simpler structure than the canonical formalism. As far as I know, this particular use of Lagrangian quantization has not been pursued.

9.8 Conclusion

At this point, one had a good understanding of the classical theory of a general nonlinear field theory whose equations of motion are derived from an action. When there is a function group as the invariance group, there are strong and weak conservation laws. The existence of the strong laws on the one hand leads to limitations on the motion of the sources of the field and on the other to constraints whether one introduces the canonical formalism or through the Lagrangian formalism. In either case, the constraints generate the invariant transformations.

One of Peter's original goals has not been realized nor is there any sense in which one thinks it may be satisfied. That is, to be able to recover quantum equations for the sources of the gravitational field in the manner of EIH. We were able to write down a phase space expression for the superpotentials [14], but there seems to be no way to express its action locally so that something like the Schrodinger equation results. On the other hand, the conservation laws are intrinsic to the quantum structure as well as the classical formalism. Therefore, quantum gravity will impose restrictions on interacting matter fields which are its sources. However, how that will appear is still unclear and may be understood only in terms of a fully unified quantum field theory. Looking for restrictions on localized matter does not seem meaningful in quantum field theory.

While a quantum theory of gravity was not completed, the main outlines of what should be kept in the transition from classical theory to quantum theory was discussed in detail. In particular, it is the constraint algebra which should be kept because the constraints are the generators of the invariant transformations. And it is the symmetries which are important in giving meaning to the quantum states. Unfortunately, without knowing something about the topology and geometry of the Hilbert space, one can only postulate that the dynamical operators should be Hermitian.

In this connection, it is worth noting that later Komar [24] suggested that the constraints need not be Hermitian. Indeed, in quantum electrodynamics only the self-dual part of the constraints need be taken as zero operators on the Hilbert space. This is essentially the way the constraints are applied in the Ashtekar formalism [25] where some progress on these questions has taken place.

Apart from laying of the groundwork for a quantum theory of gravity with the construction of the Hamiltonian and study of the constraint algebra, the most important result of this period is the recognition that the observables must commute with the constraints and therefore with the Hamiltonian. This has the strange result that the observables appear to be frozen in time. As a result, one began to think of general relativity as an *already parametrized* theory and one sought, without success, to construct a time variable from within the geometrical structures themselves.

With the more geometrical formulation of the Hamiltonian by Dirac in 1959 [26], the lapse and the shift replace $g^{0\mu}$ and are clearly identified as arbitrary elements which are not dynamical. Essentially, these quantities are the conjugates of the primary constraints. In this structure, the Hamiltonian is constructed from the secondary constraints, the scalar constraint which is quadratic in the momenta and the three-dimensional vector constraints which are linear in the momenta. The latter generate the diffeomorphisms in the space-like surface $x^0 =$ constant while the former generates the evolution off the surface. The observables, of course, must commute with all of these constraints.

While we are now 50 years later still without a quantum theory of gravity, the fundamental ideas developed in those early years by Peter Bergmann and his students continue to form the basis of current work to the extent that one doesn't mention those ideas explicitly any longer. Today, not only is work continuing on the effort to quantize the original Einstein theory, but Einstein's dream of a unified theory is being pursued in string theory and super-gravity.

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