PHY662, Spring 2004, Apr. 29, 2004

29th April 2004

1 Miscellaneous

1. We conclude today with the quantization of the electromagnetic field and its consequences.
2. The final exam is on Monday, May 3. It is scheduled for 5:30 PM, Room 202/204.
3. There will be an exam review session at 5:00 PM, Sunday, May 2, Room 202/204.

2 Quantizing the EM field

During most of this course, photons have been discussed only as parenthetical comments. Starting with magnetic resonance, we have noticed that the transitions between states take place for electromagnetic fields that oscillate with frequency such that $\hbar\omega = \Delta E$. This is true for external classical potentials, as this is the type of external potential that can lead to interference between states separated in energy by ΔE.

You of course know that light comes in waves with quantized energy steps of size $\hbar\omega$. To realize this picture, you might impose local gauge invariance of the wave function ($\psi \to e^{-iq\Lambda/\hbar c}\psi$) and maintain the Schrodinger equation with the introduction of the vector potential $\vec{A}$. This gives the interaction between $\vec{A}$ and ψ. Our observations of the electromagnetic field at macroscopic scales (the classical realm) suggests the Hamiltonian for the field $\vec{A}$ (and ϕ, but we will be assuming the Coulomb gauge where $\phi = 0$).

Today we will quickly sketch how to use the classical suggestion to create a correct quantum theory and then apply this description to spontaneous decay of an excited state in a hydrogen atom.

2.1 Classical considerations $\to a, a^\dagger$

We will be replacing the classical vector field $\vec{A}(\vec{r})$ by an operator field $\vec{A}^{(\mathrm{op})}(\vec{r})$. (Next semester you might in addition replace the field $\psi(\vec{r})$ by the operator field $\psi^{(\mathrm{op})}(\vec{r})$, so that you can consider creating and destroying fermions such as electrons.)

To summarize pp. 506-515 of Shankar:

1. From macroscopic observation and symmetry, we have a good idea of what the Lagrangian $\mathcal{L}$ is for the EM field, expressed in terms of $\vec{A}$. One then wants to determine what quantum Lagrangian or Hamiltonian this classical theory approximates.
2. To reproduce the classical equations of motion, the quantum momenta conjugate to the coordinates should obey commutation relations like $[q,p] = i\hbar$, in some form (but the "position" & "momenta" are functions of position, i.e., are fields). The classical momenta are found using $\Pi_i = \frac{\partial \mathcal{L}}{\partial \dot{A}_i}$. For the EM field, $\Pi = -\frac{\vec{E}}{4\pi c}$.
3. Shankar then discusses the constraints $\vec{\nabla} \cdot \vec{A} = 0$ and $\vec{\nabla} \cdot \vec{\Pi} = 0$, which come from the full gauge-invariant Lagrangian. These constraints are hard to directly incorporate into the canonical commutation relations.
4. *However*, if you
    (a) change coordinates to Fourier space, e.g., $\vec{A}(\vec{r}) \to \vec{A}(\vec{k})$, and
    (b) express $\vec{A}(\vec{k})$ in bases $\vec{\epsilon}(\vec{k},j)$, $j = 1,2,3$, with $\vec{\epsilon}(\vec{k},3) \parallel \vec{k}$, $\vec{A}(\vec{k}) = \sum_j a_j(\vec{k})\vec{\epsilon}(\vec{k},j)$, the classical constraints simply become $a_3 = 0$ *AND*

$$\mathcal{H} = \sum_{j=1}^{2} \int \omega [a^*(\vec{k},j) a(\vec{k},j)] \, d^3\vec{k} \, .$$

5. So we take $a(\vec{k}, j = 1,2)$ as normal coordinates for oscillators! We can carry out all the usual oscillator stuff, e.g., writing $P \propto a^\dagger + a$, $Q \propto i(a^\dagger - a)$, so that $\mathcal{H} \sim \sum \int Q^2 + P^2$, with Q and P obeying the canonical commutation relations.
6. This is just like oscillator world we wrote down last time (except for the interactions and counting the degrees of freedom). (The spectrum or $\rho(\omega)$ is also fixed for the EM field.) So, the quantum state of the field can be described by a set of states for quantum oscillators which can be described by the ladder number of each oscillator. The ladder number $a^\dagger a$ is the occupation number or number of photons in that state. Adding or subtracting a photon corresponds to the raising or lowering of a quantum oscillator.

2.2 Spontaneous decay

Let us consider the very specific problem considered in the course text, that of a transition

$$|2lm; 0_{\vec{k},j}\rangle \to |100; 1_{\vec{k},j}\rangle \, ,$$

where $2lm$ indicates a first excited state of the Coulomb (hydrogenic atom) problem (we are neglecting the spin of the electron), 100 indicates the ground state, and $n_{\vec{k},j}$ denotes n photons with momentum $\vec{k}$ and polarization $j \in \{1,2\}$.

The unperturbed Hamiltonian that describes the time evolution of ψ, with mass m, charge e, and momentum p, and the photon modes is

$$H^0 = \frac{p^2}{2m} - \frac{e^2}{r} + \sum_{\vec{k},j} \hbar c k \left[a^\dagger(\vec{k},j) a(\vec{k},j) + \frac{1}{2} \right] .$$

The perturbation is the interaction term that comes from expanding $\frac{1}{2m}(p - \frac{qA}{c})^2$, neglecting the A^2 term as smaller in magnitude: $H' = \frac{e}{m}\vec{p} \cdot \vec{A}$. Note now that $\vec{A}$ is now an operator, so that the photons enter the quantum description, and that it does not carry any explicit time dependence.

So one of the types of factors we need to compute is

$$\begin{aligned}
\langle 100; 1_{\vec{k},j} | H' | 2lm; 0_{\vec{k},j} \rangle &\propto \langle 100; 1_{\vec{k},j} | \vec{A} \cdot \vec{p} | 2lm; 0_{\vec{k},j} \rangle \\
&\propto \int d^3\vec{r}\, \psi^*_{100} \vec{\epsilon}(\vec{k},j) \cdot (-i\hbar\vec{\nabla}) \psi_{2lm} \\
&\propto \vec{\epsilon}(\vec{k},j) \cdot \int d^3\vec{r}\, \psi^*_{100} \vec{r} \psi_{2lm} \quad [\text{dipoleapprox.}].
\end{aligned}$$

As ψ_{200} and ψ^*_{100} are both even, this matrix element is zero for $l = 0$. Note that this matrix element is *zero* for a classical field $\vec{A} = 0$. This is an example of spontaneous emission.

When you simply (!) work through the integrals (averaging over $m = -1, 0-, 1$) which you can do on your own following the text, you get

$$\Gamma_{21m \to 100, \hbar\omega} = \left(\frac{2}{3}\right)^8 \alpha^5 \frac{mc^2}{\hbar} = (1.6 \times 10^{-9}\,\mathrm{s})^{-1} .$$