PHY662, Spring 2004, Apr. 27, 2004

27th April 2004

1 Miscellaneous

1. This week, we will examine the quantization of the electromagnetic field and its consequences.
2. The final exam is on Monday, May 3. It is scheduled for 5:30 PM, Room 202.
3. Reminder, homework is due Thursday.

2 Structure of the problem

Suppose we wish to understand why the sky is blue. Some understanding can be gained from a classical calculation and a simplified picture of an atom, but for a precise justification, one needs a quantized radiation field.

We will take steps towards quantum field theory (QFT), which is the subject of Quantum III, which many of you will be taking next semester. Here we will not be quantizing the matter (electron) field itself. As the electromagnetic field is a vector and has constraints, it can be tricky to justify and work with the formalism. So let's first consider a "toy model" where we can see much of the structure of the problem. This will be a one-dimensional scalar-coupling problem, simpler than the more complex coupling of an electron (possibly with spin) to the quantized EM field.

2.1 "Oscillator world"

Let the Universe be composed of harmonic oscillators. There will be a single degree of freedom, the "primary oscillator" (corresponding to the electron in an atom in the full problem) with unperturbed Hamiltonian

$$H^0 = \hbar\omega(a^\dagger a + \frac{1}{2})\,.$$

The rest of the Universe will be a large number of harmonic oscillators, the "bath" or "reservoir". We will index these oscillators by $i = 1, \ldots, N$. We will compute results for two cases: when the oscillators are classical and when they are quantum. The Hamiltonian for each oscillator is then

$$H_i \quad = \quad \begin{cases} \frac{k_i x_i^2}{2} + \frac{p_i^2}{2\mu}\,, & [\text{classical energy of bath oscillator}: x_i, p_i \text{ are classical numbers}] \\ \hbar\omega_i(a_i^\dagger a_i + \frac{1}{2}) & [\text{quantum bath}: a_i \text{ is the lowering operator for oscillator } i] \end{cases}$$

For the quantum case, the occupation number of oscillator i will be denoted by m_i. Of course, in real life, all is quantum, so the classical bath will be at best an approximation.

The unperturbed quantum Hamiltonian is

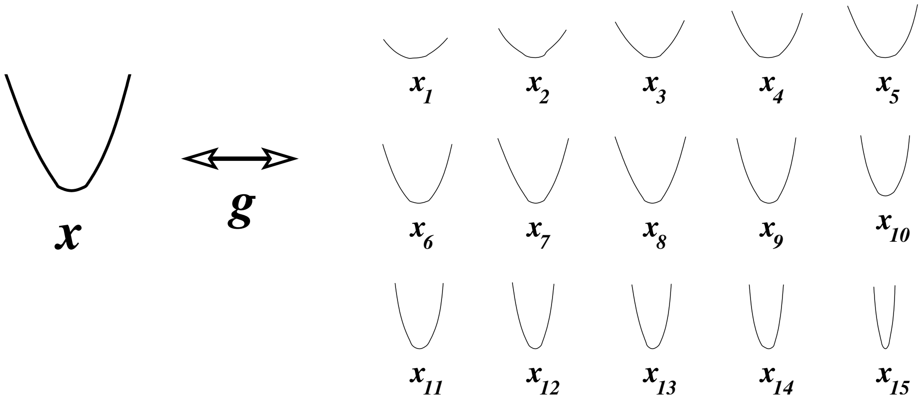
$$H^0 \quad = \quad \begin{cases} \hbar(a^\dagger a + \frac{1}{2}) & [\text{classical bath}] \\ \hbar(a^\dagger a + \frac{1}{2}) + \sum_i \hbar\omega_i(a_i^\dagger a_i + \frac{1}{2}) & [\text{quantum bath}] \end{cases}$$

So far, the dynamics and eigenstates of the full quantum problem are trivial compositions of product states. Now lets add a weak coupling between the primary oscillator and the “bath” of N oscillators. We will take the simple coupling

$$H' = \frac{g}{\sqrt{N}} x x_{tot} = \left(\frac{g}{\sqrt{N}}\right) x \sum_i x_i\,,$$

where x is the coordinate of the primary oscillator and g is the strength of the coupling between the primary oscillator and the bath oscillators. The factor of $N^{-1/2}$ will give a well-defined solution when we take the limit of large N. (N will play the role of volume V for the quantized electromagnetic field.)

We can sketch “oscillator world”, showing the potentials for each oscillator as a function of coordinate and the coupling g schematically (primary on the left, bath on the right):



[Maybe this represents an electron in a well, electrostatically interacting with a set of distant electrons, each in their own wells of varying curvatures.]

2.2 Transitions in a classical bath

Let’s look at transition rates in oscillator world, for ***classical*** bath oscillators. The interaction Hamiltonian is $gN^{-1/2}x\sum_i x_i$, where x_i are *classical numbers.*

The x_i then behave as

$$x_i = A_i \cos(\omega_i t + \beta_i) = \frac{A_i}{2}\left[e^{-i(\omega_i t+\beta_i)} + e^{i(\omega_i t+\beta_i)}\right],$$

where the A_i are the amplitudes of each oscillator. The phases β_i are determined by initial conditions and are taken to be random.

Applying Fermi's golden rule for transitions between primary oscillator states $|n_1\rangle$ and $|n_2\rangle$ gives

$$\Gamma_{n_1\to n_2} = \frac{2\pi}{\hbar}\sum_i |gN^{-1/2}\frac{A_i}{2}\langle n_2|x|n_1\rangle|^2 \delta[\omega(n_2-n_1)\pm\omega_i]\,.$$

Note that the δ-function has an argument that includes the time dependence of the external perturbation, with contributions to A_i from $e^{-i\omega t}$ and $e^{i\omega t}$. The matrix elements are straightforward:

$$\begin{aligned}\langle n_2|x|n_1\rangle &= f\langle n_2|(a+a^\dagger)|n_1\rangle \\ &= f\sqrt{n_1}\delta_{n_2+1,n_1} + f\sqrt{n_2}\delta_{n_2-1,n_1}\,,\end{aligned}$$

where $f = \left(\frac{\hbar}{2\mu\omega}\right)^{1/2}$.

Taking the limit of large N,

$$\sum_i \to N\int_0^\infty d\omega'\,\rho(\omega')\,,$$

where $N\rho(\omega')$ is the number density of oscillators in the interval $d\omega'$ in the limit of large N, so we get

$$\begin{aligned}\Gamma_{n_1\to n_2} &= \frac{2\pi f^2 g^2}{\hbar}\int_0^\infty d\omega'\rho(\omega')\delta(\omega\pm\omega')\frac{A^2(\omega')}{4}(n_1\delta_{n_2+1,n_1} + n_2\delta_{n_2-1,n_1}) \\ &= \frac{\pi f^2 g^2\rho(\omega)}{2\hbar}A^2(\omega)\left[n_1\delta_{n_2+1,n_1} + (n_1+1)\delta_{n_2-1,n_1}\right],\end{aligned}$$

by using $\delta^2_{n_2+1,n_1} = \delta_{n_2+1,n_1}$ and $\delta_{n_2+1,n_1}\delta_{n_2-1,n_1} = 0$. As the *energy* of an oscillator is , this can be rewritten as in terms of the *energy density in the bath $I(\omega)$*,

We have found the transition rate caused by a bath of classical oscillators. Note that the transition rate up and down in energy (of the primary oscillator) *are both proportional to the energy density at ω*.

2.3 Taking a quantum bath

Now let's use the quantum Hamiltonian for the oscillators. Then we *include* the bath oscillators in the state description to get basis states that are given by the quantum numbers for the primary oscillator and all of the bath oscillators:

$$|n\rangle \to |n; m_1, m_2, \ldots, m_N\rangle\,.$$

The time evolution without the interaction term is

$$|n;m_1,m_2,\ldots,m_N(t)\rangle = |n;m_1,m_2,\ldots,m_N(0)\rangle e^{-i\omega nt-i\omega_1 m_1 t-\ldots-i\omega_N m_N t}\,.$$

We can expand an *incoherent* state at time $t=0$ as

$$|\psi(0)\rangle = \sum C_{n_1} c_{1,m_1} c_{2,m_2}\ldots c_{N,m_n}|n;m_1,m_2,\ldots,m_N\rangle\,,$$

so that the energy associated with bath oscillator i is

$$\begin{aligned} E_i &= \hbar\omega_k(\sum_k c^*_{i,k}c_{i,k}k+\frac{1}{2}) \\ &= \hbar\omega_i(\langle m_i\rangle+\frac{1}{2})\,. \end{aligned}$$

The δ-function in Fermi's golden rule now ensures $E_{\text{final}} = E_{\text{initial}}$, as there is no time dependence in H' anymore. So let's calculate matrix elements only for those states. The only non-zero matrix elements that conserve energy are for $\omega_i = \omega$, with

$$\begin{aligned} |n;m_1,\ldots,m_i,\ldots,m_N\rangle &\rightarrow |n+1;m_1,\ldots,m_i-1,\ldots,m_N\rangle \\ |n;m_1,\ldots,m_i,\ldots,m_N\rangle &\rightarrow |n-1;m_1,\ldots,m_i+1,\ldots,m_N\rangle \end{aligned}$$

Non-zero matrix elements of $H' = gN^{-1/2}\sum_i f_0 f_i(a_i+a_i^\dagger)(a_0+a_0^\dagger)$ are

$$\begin{aligned} \langle n+1;m_1,\ldots,m_i-1,\ldots,m_N|H'|n;m_1,\ldots,m_i,\ldots,m_N\rangle &= gN^{-1/2}f_i f\sqrt{m_i}\sqrt{n+1} \\ \langle n-1;m_1,\ldots,m_i+1,\ldots,m_N|H'|n;m_1,\ldots,m_i,\ldots,m_N\rangle &= gN^{-1/2}f_i f\sqrt{m_i+1}\sqrt{n}\,. \end{aligned}$$

This gives the transition rates

$$\Gamma_{n_1\to n_1+1} = \frac{2\pi g^2 f^2\rho(\omega)}{\hbar}f_i^2\langle m_i\rangle(n+1)$$

and

$$\Gamma_{n_1\to n_1-1} = \frac{2\pi g^2 f^2\rho(\omega)}{\hbar}f_i^2(\langle m_i\rangle+1)n\,.$$

Note that:

1. The rate of up-transitions is the same as the classical case - proportional to the energy density of the bath, above the minimal energy, at/near $\omega_i = \omega$. ($\frac{A_i^2}{4} = \frac{E_i^{\text{classical}}}{2k} = \frac{\hbar\omega_i m_i^{\text{classical}}}{2\mu_i\omega_i^2} = f_i^2\hbar\omega_i m_i^{\text{classical}}$.)
2. The rate of down-transitions is *higher:* there is the "stimulated" transition rate proportional to energy density ***and*** the spontaneous transition rate that is independent of the occupations of the bath oscillators.

What is sometimes said is that the vacuum fluctuations in the x_i fields drive the "spontaneous" transitions. In any case, it does come from the quantum nature of the bath.