

# PHY662, Spring 2004, Apr. 13, 2004

15th April 2004

## 1 Miscellaneous

1. Reading: Continue Shankar Ch. 18 for electromagnetism, also Griffiths Ch. 9 (though it is a bit simplified, especially as it sticks to two-level systems at first). Skip higher order perturbation theory in Shankar and read up to “Field Quantization” (p. 506 in the second edition) by Tuesday, April 13. Much of what follows in today’s notes is derived from Shankar and Ch. 13 of G. Bayms’ *Lectures on Quantum Mechanics*.
2. Office hours are back to 3:30-5:00 on Monday. This week’s homework will be in your mailboxes later this afternoon and is due Tuesday, April 20.

## 2 Electromagnetic waves

Remember the wave equation for the vector potential:

$$\nabla^2 \vec{A} - \frac{1}{c^2} \frac{\partial^2 \vec{A}}{\partial t^2} = 0.$$

These equations give that waves in  $\vec{A}$  travel at speed  $c$  and that plane wave solutions for  $\vec{A}$  are of the form

$$\vec{A} = \vec{A}_0 \cos(\vec{k} \cdot \vec{r} - \omega t)$$

with  $k^2 c^2 = \omega^2$  and with the important resulting condition from the Coulomb gauge,  $\nabla \cdot \vec{A} = 0$  (transverse waves)

$$\vec{k} \cdot \vec{A}_0 = 0.$$

### 2.1 Aharonov-Bohm effect

Clarification from last lecture: one can set  $\vec{A} = 0$  over *any region where  $\vec{B} = 0$*  by carrying out a gauge transformation with  $\nabla \Lambda = -\vec{A}$ . This gauge change modifies the wave function at  $\vec{r}_1$  by a well-defined phase change,

$$\psi(\vec{r}_1) \rightarrow e^{-i(q/\hbar c) \int_{\vec{r}_0}^{\vec{r}_1} \vec{A} \cdot d\vec{s}} \psi(\vec{r}_1).$$

There are *no observable effects within this region* from magnetic fields.

But relative changes in phases along two paths can lead to changes in interference: a quantum particle can explore more than one path simultaneously, so changes in  $\vec{A}$  can lead to observable effects!

If we compare the phase change due to  $\vec{A}$  for two paths of a particle about a solenoid, we can see this interference effect, whenever the flux is *not* a multiple of the flux quantum  $\frac{hc}{q}$ .

This interference effect has applications in imaging. Tonomura's group at Hitachi lab has imaged vortices of magnetic flux in superconductors using this effect. Here is an example - electrons that pass through a superconductor (niobium) subject to a magnetic field interfere with electrons that pass by the superconductor. This gives an interference pattern / hologram that can be used to infer the magnetic field in the superconductor. Note that the field is not uniform!

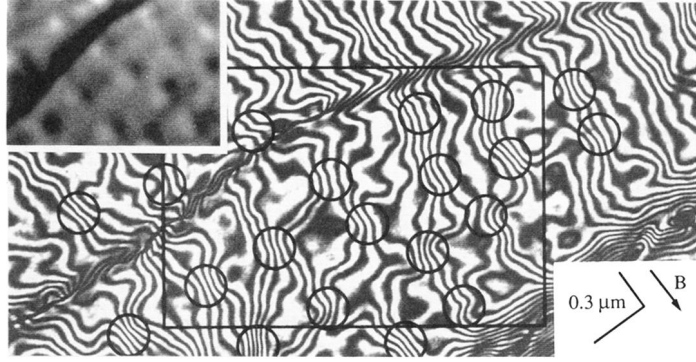


FIG. 2. Interference micrograph of a vortex lattice (phase amplified 16 $\times$ ). Projected magnetic lines of force are directly observed as contour fringes. They are concentrated locally at the circled regions, becoming narrowly spaced. These regions spatially coincide with the spots observed in the Lorentz micrograph (inset), and are identified to be quantized vortices. A bend contour runs diagonally through the Nb foil.

This sensitivity to magnetic fields is used in SQUID (Superconducting Quantum Inteference Device) or even for electrons in mesoscopic devices.

## 2.2 Electromagnetic modes

We can simply express solutions to the wave equation for  $\vec{A}$  using plane waves in a box of volume  $V$  as

$$\vec{A}(\vec{r}, t) = \sum_{\vec{k}, \vec{\lambda}} \frac{1}{\sqrt{V}} \left[ A_{\vec{k}, \vec{\lambda}} \vec{\lambda}(\vec{k}) e^{i(\vec{k} \cdot \vec{r} - \omega t)} + A_{\vec{k}, \vec{\lambda}}^* \vec{\lambda}^*(\vec{k}) e^{-i(\vec{k} \cdot \vec{r} - \omega t)} \right],$$

where the second term is the c.c. of the first to ensure that  $\vec{A}$  is real (which it must be, in order for particle conservation to hold and for  $\vec{B}$  to be real) and the  $V^{-1/2}$  factor is a convenient normalization. The Coulomb gauge condition  $\vec{\nabla} \cdot \vec{A} = 0$  is satisfied iff  $\vec{k} \cdot \vec{\lambda} = 0$ . The polarization vectors  $\vec{\lambda}$  can be complex, but one often chooses two plane polarizations, with  $\vec{\lambda}_{1,2} \perp \vec{k}$ ,  $\vec{\lambda}_1 \perp \vec{\lambda}_2$ . The scalar  $A_{\vec{k}, \vec{\lambda}}$  gives the amplitude and phase of the wave.

The total electromagnetic energy in the volume  $V$  is

$$E = \sum_{\vec{k}, \vec{\lambda}} \frac{\omega^2}{2\pi c^2} |A_{\vec{k}\vec{\lambda}}|^2.$$

### 3 Calculating electromagnetic transition rates

So let's imagine a single-electron atom in its ground state and bathed in incoherent radiation, such as from a typical black body or vapor lamp. How long does the atom spend in the ground state  $|0\rangle$  before it is excited to a given state  $|n\rangle$  of higher energy?

Applying Fermi's golden rule gives

$$\Gamma_{0 \rightarrow n} = \frac{2\pi}{\hbar} |\langle n | \left( \frac{-e}{mc} \right) \vec{A} \cdot \vec{p} | 0 \rangle|^2 \delta(E_n - E_0 - \hbar\omega),$$

where the interaction term that is proportional to  $A^2$  in the interaction Hamiltonian has been dropped, taking the intensity of the incident radiation small compared to the electric field due to the nucleus of the atom and we are working in the Coulomb gauge,  $\vec{A} \cdot \vec{p} = \vec{p} \cdot \vec{A}$ .

For light atoms, transitions involve photons with wavelengths of the order of 100's of nm, much larger than the atomic size of order 0.1 nm, so it is fair to write  $A_{\vec{k}\vec{\lambda}} e^{-i\vec{k} \cdot \vec{r}}$  just as  $A_{\vec{k}\vec{\lambda}}$  (in a moment we will see why this is referred to as the electric dipole approximation).

Writing this as a sum over incoherent positive-frequency modes (since absorption) gives

$$\Gamma_{0 \rightarrow n} = \frac{2\pi}{\hbar} \sum_{\vec{k}, \vec{\lambda}} V^{-1} |A_{\vec{k}\vec{\lambda}}|^2 |\langle n | \left( \frac{-e}{mc} \right) \vec{\lambda} \cdot \vec{p} | 0 \rangle|^2 \delta(E_n - E_0 - \hbar\omega).$$

Now consider the matrix element  $\langle n | \vec{p} | 0 \rangle$ . Using  $\vec{p}/m = (i\hbar)^{-1} [\vec{r}, H_0]$ ,

$$\begin{aligned} \langle n | \vec{p} | 0 \rangle &= \frac{m}{i\hbar} \langle n | (\vec{r} H_0 - H_0 \vec{r}) | 0 \rangle \\ &= \frac{m}{i\hbar} (E_0 - E_n) \langle n | \vec{r} | 0 \rangle. \end{aligned}$$

Using  $(E_n - E_0) = \hbar\omega$ , we can now write

$$\Gamma_{0 \rightarrow n} = \frac{2\pi e^2}{\hbar c^2} \sum_{\vec{k}, \vec{\lambda}} \omega^2 V^{-1} |A_{\vec{k}\vec{\lambda}}|^2 |\lambda \cdot \langle n | \vec{r} | 0 \rangle|^2 \delta(E_n - E_0 - \hbar\omega).$$

To carry out the calculation further, we need to work with the sum  $\sum_{\vec{k}, \vec{\lambda}}$ , to convert it into an integral over energy. As the dispersion relation for light is rather simple,  $\omega = ck$ ,

$$\sum_{\vec{k}} \rightarrow \left( \frac{L}{2\pi} \right)^3 \int d^3k = V \int \frac{k^2 dk d\Omega}{(2\pi)^3} = V \int \frac{\omega^2 d\omega d\Omega}{(2\pi c)^3},$$

we get

$$\begin{aligned}\Gamma_{0 \rightarrow n} &= \frac{2\pi e^2}{\hbar c^2 (2\pi c)^3} \int d\omega d\Omega \omega^4 |A_{\vec{k}\vec{\lambda}}|^2 |\lambda \cdot \langle n | \vec{r} | 0 \rangle|^2 \delta(E_n - E_0 - \hbar\omega) \\ &= \frac{2\pi e^2 \omega^4}{\hbar^2 c^2 (2\pi c)^3} \int d\Omega |A_{|\vec{k}|=\omega/c, \vec{\lambda}}|^2 |\lambda \cdot \langle n | \vec{r} | 0 \rangle|^2.\end{aligned}$$

Consider radiation incident upon the atom from an incoherent polarized source with an intensity measured in energy per unit solid angle per frequency interval,  $I(\omega)$ . It turns out that

$$I(\omega) = \frac{d\Omega \omega^4 |A_{k\lambda}|^2}{(2\pi c)^4}.$$

Substituting this in gives

$$\Gamma_{0 \rightarrow n} = \frac{2\pi e^2 \omega^4}{\hbar c^2 (2\pi c)^3} (2\pi c)^4 \omega^{-4} I(\omega) |\lambda \cdot \langle n | \vec{r} | 0 \rangle|^2.$$

*What can we infer from this form of the rate?* [Can work with total position operator  $\sum r = R$ . Note that  $[L_z, R_z] = 0$ ,  $[L_z, R_x] = i\hbar R_y$ , to get selection rules for  $m'$ ,  $m$  and can also get  $l' = l \pm 1$ .]

[Note: this type of calculation is from Baym. Shankar considers the photoelectric effect, where the final state is a plane wave state. There, you also have to be careful in the initial state - it is easiest for an  $s$ -state. In this case, the matrix element is

$$\begin{aligned}\left(\frac{e}{2mc}\right) \left(\frac{1}{2\pi\hbar}\right)^{3/2} (\pi a_0^3)^{-1/2} \int d^3\vec{r} e^{-i\vec{p}_f \cdot \vec{r} / \hbar} \vec{A}_0 \cdot (-i\hbar \nabla) e^{-r/a_0} &= N \vec{A}_0 \cdot \vec{p}_f \int d^3\vec{r} e^{-i\vec{p}_f \cdot \vec{r} - r/a_0} \\ &= \frac{8\pi a_0 N (\vec{A}_0 \cdot \vec{p}_f)}{[(1/a_0)^2 + (p_f/\hbar)^2]^2}\end{aligned}$$

and inserting into the golden rule gives

$$\Gamma = \frac{2\pi}{\hbar} N^2 (\vec{A}_0 \cdot \vec{p}_f)^2 \frac{64\pi^2 a_0^6}{[1 + (p_f a_0/\hbar)^2]^4} \delta(E_f - E_i - \hbar\omega)$$

and the transition rate into the angle  $d\Omega$  of

$$\Gamma_{i \rightarrow d\Omega} = \frac{4a_0^3 e^2 p_f |\vec{A}_0 \cdot \vec{p}_f|^2}{m\pi \hbar^4 c^2 [1 + (p_f a_0/\hbar)^2]^4} d\Omega.$$

This is the transition rate for the photoelectric effect.