PHY662, Spring 2004, Apr. 1, 2004

1st April 2004

1 Miscellaneous

1. Reading: Continue Shankar Ch. 18 for time-dependent perturbation theory, electromagnetism, also Griffiths Ch. 9.
2. Time-dependent perturbation theory, especially periodic perturbations.
3. Homework will be handed out by Friday and will be due next Thursday (the next two weeks' homework will be due on Thursday). Office hours will start at 4:00 on Wednesday.

2 Fermi's Golden rule

From the derivation in last class's notes, we obtained **Fermi's golden rule**.
For $H' = V(\vec{r}) \int d\omega\, \rho(\omega) \cos(\omega t)$, the transition *rate* from $i \to f$ is

$$\Gamma \approx \frac{\pi}{2\hbar^2} |\langle f|V(\vec{r}, \omega_{fi})|i\rangle|^2 [\rho(\omega_{fi}) + \rho(-\omega_{fi})] \,.$$

For $H' = V(\vec{r}) e^{-i\omega t}$, the transition *rate* from $i \to f$ is

$$\Gamma \approx \frac{2\pi}{\hbar} |\langle f|V(\vec{r}, \omega)|i\rangle|^2 \delta(E_f - E_i - \hbar\omega) \,.$$

In this form, the δ-function has been pulled out for convolving with $\rho(\omega)$ and the perturbation is taken to be of the form $V(\vec{r}, \omega) e^{-i\omega t}$.A $\cos(\omega t)$ perturbation is the sum of two exponential perturbations with opposite frequencies.

The latter, more common form, expresses the rule in a way that allows one to integrate over either E_f or over ω. It can be used to obtain the other form by writing $\cos(\omega t) = \frac{1}{2}\left(e^{i\omega t} + e^{-i\omega t}\right)$.

The result is that the transition rate is proportional to the square of the matrix element between the initial and final states of the spatial component of the perturbation. This is a first order calculation in the amplitudes, with V being the part of the total Hamiltonian $H = H^0 + H'$ that "causes a transition" between $|i\rangle$ and $|f\rangle$. This just comes from the rate of change of the wave function including a part from H', as per the Schrodinger equation. The δ-function is a result of energy conservation, with one quantum of the perturbing potential having an energy $\hbar\omega$.

This energy conservation comes from an integral over time of an exponential with an imaginary argument. There are three parts to this argument: the frequencies of the initial and final state and the frequency from the perturbation. When the sum of these is zero, integrating over long times gives the δ-function.

3 Examples

Fermi's golden rule allows one to compute real transition rates for real atoms. To do so, we will need to build up some background with electromagnetism as applied in quantum mechanics. Before we do that, let us try a more artificial example.

Suppose you have a charged particle in a 1D harmonic oscillator with frequency ω_0. You modify the potential by applying a classical electric field $\mathcal{E}(t)$ that is uniform in space and that behaves like "white noise" over time. At any given instant t, the change in potential is $-q\mathcal{E}(t)X$. White noise has the property that the fourier component at each frequency is uniform. Writing $\mathcal{E}(t) = \int_0^\infty d\omega\, E_0 \cos(\omega t)\rho_0$ gives

$$\Delta V(t) = q\mathcal{E}(t)X = \int_0^\infty d\omega\, qXE_0\rho_0 \cos(\omega t)\,,$$

where it is understood that there is a randomness between the frequencies that makes the perturbation incoherent (one could add a random phase to each ωt). Fermi's golden rule then gives the transition rate from the ground state to the first excited state as

$$\begin{aligned}
\Gamma_{0\to1} &= \frac{\pi}{2\hbar^2}|\langle 1|qXE_0|0\rangle|^2\rho_0 \\
&= \frac{\pi q^2 E_0^2}{2\hbar^2}|\langle 1|\left(\frac{2\hbar}{m\omega_0}\right)^{1/2}(a+a^\dagger)|0\rangle|^2\rho_0 \\
&= \frac{\pi q^2 E_0^2 \rho_0}{m\omega_0\hbar}\,.
\end{aligned}$$

(Let us check the units to help us believe that the calculation was properly done.)

4 Electromagnetism

[See Shankar for Maxwell's equation and this discussion.]

The highlights are: rewriting the electromagnetic field using the potentials $\vec{A}$ and ϕ,

$$\begin{aligned}
\vec{B} &= \nabla\times\vec{A} \\
\vec{E} &= -\frac{1}{c}\frac{\partial\vec{A}}{\partial t} - \nabla\phi\,;
\end{aligned}$$

the choice of Coulomb gauge for the "free electromagnetic field" (where sources density $\rho = 0$ and current $\vec{j} = 0$)

$$\begin{aligned} \nabla \cdot \vec{A} &= 0 \\ \phi &= 0\,; \end{aligned}$$

and the equations of motion for $\vec{A}$ in the Coulomb gauge

$$\nabla^2 \vec{A} - \frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2} = 0\,.$$

These equations give that waves in $\vec{A}$ travel at speed c and that plane wave solutions for $\vec{A}$ are of the form

$$\vec{A} = \vec{A}_0 \cos(\vec{k} \cdot \vec{r} - \omega t)$$

with the important resulting condition (transverse waves)

$$\vec{k} \cdot \vec{A}_0 = 0\,.$$

The resulting $\vec{E}$ and $\vec{B}$ fields have equal magnitude and the energy density is

$$u = \frac{1}{8\pi}(|\vec{E}|^2 + |\vec{B}|^2)\,.$$

4.1 Potentials in quantum theory

Shankar works using path integrals. This is important and I suggest you read it, but we are not focusing on path integrals this term. Remember that the Hamiltonian for a charged particle in only an electromagnetic potential is

$$\frac{\hbar^2}{2m}\left(\vec{p} - \frac{q}{c}\vec{A}\right)^2 + q\phi\,.$$

Let's rederive the conservation of current using this Hamiltonian.