PHY662, Spring 2004, Mar. 18, 2004

18th March 2004

1 Miscellaneous

1. Continue Shankar Ch. 18 for time-dependent perturbation theory.
2. I am out of town next week (exam on Tuesday, we will schedule a makeup class for Thursday).
3. Exam review session on Sunday, 1 PM, Room 204.
4. Today:
   (a) Continue time-dependent perturbation theory.

2 Time-dependent perturbation theory

> "You have to wonder how this happened to him," Lundergaard said. "Was he calculating the transition amplitudes between the unperturbed eigenstates due to the presence of the perturbation in order to determine transition probabilities in time-dependent quantum phenomena, and the next day, strapping a TV antenna to his head?" [From the 17 March 2004 issue of *The Onion*.]

We are solving for the time dependence of $\psi(t)$, with $i\hbar\frac{\partial}{\partial t}\psi(t) = H\psi(t)$, where $H(t) = H^0 + H'(t)$.

Our approach *will* be to compute transition amplitudes between the unperturbed eigenstates, where the unperturbed eigenstates $|n\rangle$ are the eigenvectors of H^0. By expanding $|\psi(t)\rangle = \sum_n c_n(t)|n\rangle$ and changing to the coefficients $d_n(t) = c_n(t)e^{iE_nt/\hbar}$, we derived the exact result

$$i\hbar\dot{d}_f = \sum_n d_n(t)\langle f|H'|n\rangle e^{i\omega_{fm}t}\,.$$

This equation gives the rate of change of the amplitude to be in a final state $|f\rangle$ that is the sum over transition amplitudes from n to f that are equal to the matrix elements of H'. We say that the non-zero matrix elements of H' cause transitions between the unperturbed eigenstates.

Griffiths carries out this derivation for the more specific case of a 2×2 Hamiltonian matrix.

2.1 Zeroth order

To lowest order, $\dot{d}_n = 0$. Let's take the initial condition that $\psi(0)$ is equal to some eigenstate $|i\rangle$. Then $d_f = \delta_{fi}$ to zeroth order.

2.2 First order

The first order result is found by putting in the zeroth order solution for d_n on the right hand side of the equation at the end of Sec. 4.1. This gives

$$i\hbar \dot{d}_f = \langle f|H'|i\rangle e^{i\omega_{fi}t} \,,$$

which has solutions

$$d_f = \delta_{fi} - \frac{i}{\hbar}\int_0^t dt' \, \langle f|H'|i\rangle e^{i\omega_{fi}t} \,.$$

This is the important first order result that will get us going on several topics.

2.3 Types of perturbations

2.3.1 Direct computation

The first-order formula for d_f can be directly applied, as Shankar does in Eqns. (18.2.10) through (18.2.13): take a harmonic oscillator with $H^0 = \hbar\omega(a^\dagger a + \frac{1}{2})$ and a time-dependent perturbation $-e\mathcal{E}Xe^{-t^2/\tau^2}$, $X = \left(\frac{\hbar}{2m\omega}\right)^{1/2}(a + a^\dagger)$. Then

$$P_{0\to1} = |d_1(\infty)|^2 = \left| \frac{ie\mathcal{E}}{\hbar}\left(\frac{\hbar}{2m\omega}\right)^{1/2} \int_{-\infty}^{\infty} e^{-t/\tau^2} e^{i\omega t} \right|^2 = \frac{e^2\mathcal{E}^2\pi\tau^2}{2m\omega\hbar} e^{-\omega^2\tau^2/2} \,.$$

What if $\tau \to \infty$?

2.3.2 Sudden Perturbations

Shankar has a section on sudden perturbations that actually has several types of changes in the Hamiltonian:

- A brief, *finite* H': in the limit that the duration of the perturbation is short, there will be no change in $\psi(t)$. This results from the fact that for finite total H, the change in ψ over an interval of duration ϵ is zero as $\epsilon \to 0$.

- A brief *"infinite"* H'. Consider a δ-function over time change in H, with $H'(t) = \delta(t - t_0)\Delta H$. Then consider the interval $[t - \epsilon, t + \epsilon]$:

$$\begin{aligned}
\psi(t_0 + \epsilon) &= \psi(t_0 - \epsilon) + \int_{t-\epsilon}^{t+\epsilon} dt' \, \frac{\partial \psi(t')}{\partial t'} \\
&= \psi(t_0 - \epsilon) + \frac{1}{i\hbar} \int_{t-\epsilon}^{t+\epsilon} dt' \, H(t')\psi(t') \\
&= \psi(t_0 - \epsilon) + \frac{1}{i\hbar} (\Delta H) \psi(t_0) \, .
\end{aligned}$$

  This is not clearly solvable in general, as $\psi(t_0)$ is not well defined ($\psi(t)$ is changing very rapidly). Taking the perturbation to be $small$ however, allows us to assume to first order that $\psi(t)$ does not change much near t_0. The amplitude to be in a state $|f\rangle$ at $t > t_0$ is then

$$d_f = \frac{1}{i\hbar} \langle f | \Delta H | i \rangle \, .$$

- A *sudden* "permanent" change in H at a time t_0. That is, there are two Hamiltonians, $H^- = H(t < 0)$ and $H^+ = H(t > 0)$. For this section, one might assume you can solve for the eigenstates of both H^+ and H^-. Again, as the Schrodinger equation is the type of differential equation it is, over a short time interval, $\psi(t)$ will not change for finite $H^{+/-}$, so $\psi(t_0 + \epsilon) = \psi(t_0 - \epsilon)$. For example, if a system is in its ground state for $t < t_0$ and the Hamiltonian is *rapidly* changed (much faster than the frequencies in the original H), the probability that the system will be in the ground state of the new Hamiltonian is simply

$$|\langle \psi_0^+ | \psi_0^- \rangle|^2 \, ,$$

  where ψ_0^- is the ground state wave function for $t < t_0$ and $\psi_0^+$is the ground state for the Hamiltonian when $t > t_0$.
  NOTE: this is the exact solution for this case. If $H^- = H^0$ is solvable and $H^+ = H^0 + H'$ is not solvable, one can use time-dependent perturbation theory to address this problem approximately, as in Fermi's golden rule.

2.3.3 Adiabatic perturbations

Adiabatic perturbations are the limit where the changes in $H(t)$ become *very* slow. Here it is not neccesarily useful to consider $d_n(t)$ referred to a given H^0 (I miswrote in the notes for last time's class). Instead, it is useful to consider an evolving set of basis functions. Let $\psi_n(t)$ be the time-independent eigenstates for $H(t)$ (i.e., $H_n(t)\psi_n(t) = E_n(t)\psi_n(t)$ at each time t), with the eigenvalues ordered in n. When the conditions of the adiabatic theorem apply, if the system is in an eigenstate $\psi_n(t_0)$ at a given time t_0, it remains in the continuously connected eigenstate $\psi_n(t)$ at later times t (in the limit of slowly varying $H(t)$)! For example, suppose a particle is in a box and is initially in the ground state. If the walls (boundary conditions) are moved very slowly compared to the characteristic velocity of the particle, then the particle will *remain in the ground state for all times*. Note that we are not saying what the *phase* of the state is - that is more complicated, involving the dynamic phase and Berry's phase.

Here is a derivation, adapted from Griffith's discussion in Ch. 10:

Let's consider an eigenstate $|n\rangle$ for H^0 at $t = -\infty$.

We start with a <u>*small*</u> adiababtic perturbation, where H' is "small" in

$$H(t) = \begin{cases} H^0, & t < 0 \\ H^0 + \left(\frac{t}{T}\right) H', & 0 < t < T \\ H^0 + H' & T < t \end{cases} .$$

From time-*independent* perturbation theory, we can compute the first order correction to the wave function for the Hamiltonian $H^0 + H'$:

$$|n\rangle + |n\rangle^1 \quad = \quad |n\rangle + \sum_{m \neq n} \frac{\langle m|H'|n\rangle}{E_n^0 - E_m^0} |m\rangle \, .$$

We will compare this with the evolving wavefunction from time-*dependent* perturbation theory, where for $m \neq n$

$$\begin{aligned} d_m(T) &= -\frac{i}{\hbar} \int_{-\infty}^{T} dt' \, \langle m| \frac{t'}{T} H' |n\rangle e^{i\omega_{mn} t'} \\ &= -\frac{i\langle m|H'|n\rangle}{\hbar T} \int_0^t dt' \, t' e^{i\omega_{mn} t'} \\ &= -\frac{i\langle m|H'|n\rangle}{\hbar T} \left[\frac{e^{i\omega_{mn} T}}{i\omega} - \frac{(e^{i\omega T} - 1)}{i\omega T} \right] \\ &\approx \frac{-\langle m|H'|n\rangle}{E_m^0 - E_n^0} e^{i\omega_{mn} T} \end{aligned}$$

(to leading order in $1/T$). Converting back to $c_m(t)$, we get

$$c_m(T) = \frac{\langle m|H'|n\rangle}{E_n^0 - E_m^0} e^{iE_n^0 T/\hbar}$$

The time evolution gives the same coefficients as found from the time-independent determination of the nth eigenstate, up to *first* order in H', with just a phase factor. All corrections to this equality are at least second order.

NOW, we can add together a bunch of small changes. If we take N small changes of size $H' = \frac{V}{N}$, we get a total correction that is bounded by

$$N \left(\frac{V}{N} \right)^2 \to 0$$

as $N \to \infty$ (lots of small, slow changes, but totalling to a finite amount).

This shows that even large changes maintain the eigenstate as H changes, as long as the change is slow.

NOTE: Remember turning a spin with a chain of Stern-Gerlach apparatuses? How is this related?