PHY662, Spring 2004, Feb. 10, 2004

24th February 2004

1 Miscellaneous

1. HWK #7 is due Tues., Mar. 2.
2. I am out of town on Mon., Mar. 1 - so I encourage you to start early on the homework.
3. Continue to read Ch. 17 Shankar (or Griffiths Ch. 6).
4. Today: HWK #6 key, WKB wrap-up, tunneling, intro to perturbation theory in physics, maybe start non-degenerate time-independent perturbation theory.

2 Homework #6 review.

3 WKB - a little more.

1. The expansion condition is $k' \ll k^2$. This is the same as $\frac{d}{dx}(\ln k) < k$: the *fractional* rate of change of k is small over one wavelength ($\lambda = \frac{2\pi}{k}$).
2. The trickiest part of the WKB approximation is where the classical momentum vanishes. This is where the oscillating solution needs to be connected to the exponential solution. Leads to connection functions:
   (a) These are simple if the potential changes very rapidly (over a distance much less than the particle "wavelength"). Apply continuity or vanishing of the wave function.
   (b) Example: quantization condition in a well with sharp sides gives $2\int_{x_1}^{x_2} dx\, \sqrt{2m(E-V)} = nh$. [come back to after Merzbacher].
   (c) Otherwise, need to be careful, use connection formula. These give, for example, the quantization condition $2\int_{x_1}^{x_2} dx\, \sqrt{2m(E-V)} = (n+\frac{1}{2})h$ for a bound state with classical turning points x_1 and x_2.
3. Griffith's example of tunneling through a large sharp barrier.
4. Discuss tunneling out of a box.

3.1 Airy functions

These functions, $Ai(z)$ and $Bi(z)$ solve $\frac{d^2y}{dz^2} = zy$. This function has two nice uses:

1. It allows us to patch together oscillatory and damped solutions at classical turning points.
2. We can explicitly solve problems with linear potentials.

Note the form of the Airy function with a sketch.

Also note the asymptotic forms

$$Ai(z) \sim (2\sqrt{\pi} z^{1/4})^{-1} e^{-\frac{2}{3} z^{3/2}} \qquad z \gg 0$$
$$Ai(z) \sim [(\sqrt{\pi}(-z)^{1/4}]^{-1} \sin\left[\frac{2}{3}(-z)^{3/2} + \frac{\pi}{4}\right] \qquad z \ll 0\,.$$

Apply this to connection formula, following Griffiths, but in more of a sketch: the result is that

$$\psi(x) \quad \approx \quad \begin{cases} \frac{2D}{\sqrt{k(x)}} \sin[\int_x^{x_r} k(x')\, dx'], & \text{if } x < x_r \\ \frac{D}{\sqrt{k(x)}} e^{-\hbar^{-1} \int_{x_r}^{x} |k(x')|\, dx'}, & \text{if } x > x_r. \end{cases}$$

For an infinite wall at $x = 0$, this gives a quantization condition:

$$\int_0^{x_r} k(x)\, dx = (n - \frac{1}{4})\pi, \quad n = 1, 2, \ldots .$$

If the left turning point at x_l is also "soft", one gets the quantization condition

$$\int_{x_l}^{x_r} k(x)\, dx = (n - \frac{1}{2})\pi, \quad n = 1, 2, \ldots .$$

4 Perturbation theory

Again, lots of problems are not exactly solvable. The step after an attempt at an exact solution is often perturbation theory, which means looking for a nearby solution to a known solution.

This approach has been very successful for lots of problems. Here are two drawbacks:

1. You need a small parameter. Without an expansion in some variable that is "small", you are lost.
2. Many expansions are only asymptotically correct. So you require some more information or techniques to get good approximations.

Physicists have been very creative in finding expansion parameters. Here are some:

- Ratios of energies: the interaction between the magnetic dipoles in an H atom are much smaller than the Coulombic energies.
- Ratios of length scales: for example, in WKB, the length scale over which the wavelength changes is taken to be much longer than the wavelength itself.
- Dimensionality: an ϵ-expansion is often used, where the dimensionality is, say, $4-\epsilon$ or $2+\epsilon$, as many theories are solvable in 2 or 4 dimensions (how inconvenient it is that we have three space dimensions).
- Magnitude of dimensionless constants: e.g., $\alpha \approx 137.04^{-1}$.

Here is an example of point 2 above: let's compute an expansion in λ for

$$C(\lambda) = \int_{-\infty}^{\infty} e^{-x^2-\lambda x^4} \ .$$

The answer we will arrive at is

$$C(\lambda) = \sqrt{\pi} \sum_{k=0}^{\infty} \frac{(2k+1)!!}{k!} \left(\frac{\lambda}{4}\right)^k \ .$$

How does this series behave? Well only in the limit $\lambda \to 0$. (There is a clue to a problem here: when $\lambda < 0$, $C(\lambda)$ diverges.)

5 Non-degenerate Time-Independent Perturbation Theory

Follow the usual derivation - using an expansion coefficient λ rather than Shankar's parameter-free description.