

PHY662, Spring 2004, Feb. 10, 2004

19th February 2004

1 Miscellaneous

1. HWK #6 not quite complete.
2. Read Ch. 8, Griffiths.

2 WKB - semiclassical approximation

1. A method (mostly) for 1D problems. It is very useful for understanding tunneling.
2. When $E > V(x)$, the Schrodinger equation has wave type solutions, $e^{\pm ikx}$, with $k(x) = \sqrt{\frac{2m}{\hbar^2}(E - V)}$. When $E < V(x)$, the solutions are of the form $e^{\pm \kappa x}$, with $\kappa = \sqrt{\frac{2m}{\hbar^2}(V - E)}$. This is strictly true for uniform potentials $V(x) = \text{const}$.
3. Now apply a type of perturbation expansion. Formulated either as an expansion in \hbar or that the classical momentum changes slowly compared to the particle's wavelength (almost everywhere).
4. The trickiest part of this approximation is where the classical momentum vanishes. This is where the oscillating solution needs to be connected to the exponential solution. Leads to connection functions:
 - (a) These are simple if the potential changes very rapidly (over a distance much less than the particle "wavelength"). Apply continuity or vanishing of the wave function.
 - (b) Example: quantization condition in a well with sharp sides gives $2 \int_{x_1}^{x_2} dx \sqrt{2m(E - V)} = nh$. [come back to after Merzbacher].
 - (c) Otherwise, need to be careful, use connection formula. These give, for example, the quantization condition $2 \int_{x_1}^{x_2} dx \sqrt{2m(E - V)} = (n + \frac{1}{2})h$ for a bound state with classical turning points x_1 and x_2 .

Now for a sketch of the details (initial part follows Merzbacher):

Take

$$\psi(x) = \exp[i\phi(x)/\hbar].$$

Then

$$i\frac{d^2\phi}{dx^2} - \left(\frac{du}{dx}\right)^2 + [k(x)]^2 = 0$$

where

$$k(x) = \sqrt{\frac{2m}{\hbar^2} [E - V(x)]}.$$

Write

$$\left(\frac{d\phi}{dx}\right)^2 = k^2(x) + i\frac{d^2\phi}{dx^2},$$

then we can solve for $\phi(x)$ by integrating $\sqrt{k^2 + i\phi''}$. One can expand this root when $|k'| \ll k^2$. What is the physical meaning of this?

In any case, one gets the result

$$\psi(x) \approx \frac{1}{\sqrt{k(x)}} e^{\pm i \int^x dx k(x)}.$$

2.1 Airy functions

These functions, $Ai(z)$ and $Bi(z)$ solve $\frac{d^2y}{dz^2} = zy$. This function has two nice uses:

1. It allows us to patch together oscillatory and damped solutions at classical turning points.
2. We can explicitly solve problems with linear potentials.

Note the form of the Airy function with a sketch.

Also note the asymptotic forms

$$\begin{aligned} Ai(z) &\sim (2\sqrt{\pi}z^{1/4})^{-1} e^{-\frac{2}{3}z^{3/2}} & z \gg 0 \\ Ai(z) &\sim [(\sqrt{\pi}(-z)^{1/4})^{-1} \sin\left[\frac{2}{3}(-z)^{3/2} + \frac{\pi}{4}\right]] & z \ll 0. \end{aligned}$$