PHY662, Spring 2004
Examples for Adding Angular Momentum

6th February 2004

The ideas you need for the homework have been presented in lecture and/or in Shankar's text, but here are a couple of clarifying comments. All of this is standard material, which can also be found in the texts on QM on reserve in the library.

Looked at mathematically, *Clebsch-Gordon coefficients represent the transformation matrix from the product representation to a representation where the states are eigenvectors of J^2 and J_z.*

Important relations

Let $|jm\rangle$ be an eigenstate of J^2 and J_z, with $J = j\hbar$ (i.e., $J^2 = j(j+1)\hbar^2$) and $J_z = m\hbar$. Now note that $J_+J_- + J_-J_+ = (J_x + iJ_y)(J_x - iJ_y) + (J_x - iJ_y)(J_x + iJ_y) = 2J_x^2 + 2J_y^2$, as the cross terms all cancel. But in the commuttor, the squares cancel and the cross terms add up, so $[J_+, J_-] = 2(iJ_yJ_x - iJ_xJ_y) = -2i\hbar[J_x, J_y] = 2\hbar J_z$, so $J_+J_- = J_-J_+ + 2\hbar J_z$. Then

$$\begin{aligned}
J_zJ_-|jm\rangle &= J_z(J_x - iJ_y)|jm\rangle \\
&= (J_zJ_x - iJ_zJ_y)|jm\rangle \\
&= (J_xJ_z - (-i\hbar J_y) - iJ_yJ_z - i(-i\hbar J_x))|jm\rangle \\
&= J_-(J_z - \hbar)|jm\rangle \\
&= J_-(m-1)\hbar|jm\rangle \\
&= [(m-1)\hbar]J_-|jm\rangle\,,
\end{aligned}$$

and

$$\begin{aligned}
J^2J_-|jm\rangle &= (J_z^2 + J_x^2 + J_y^2)(J_x - iJ_y)|jm\rangle \\
&= (J_z^2 + \frac{1}{2}(J_+J_- + J_-J_+))J_-|jm\rangle \\
&= \{J_zJ_-(J_z - \hbar) + \frac{1}{2}(J_-J_+J_- + 2\hbar J_zJ_- + J_-J_-J_+ + 2\hbar J_-J_z)\}|jm\rangle \\
&= \{J_-(J_z - \hbar)^2 + \frac{1}{2}J_-(J_+J_- + 2\hbar J_-(J_z - \hbar) + J_-J_+ + 2\hbar J_z)\}|jm\rangle
\end{aligned}$$

$$
\begin{aligned}
&= J_-\{J_z^2 - 2\hbar J_z + \hbar^2 + \frac{1}{2}(J_+J_- + J_-J_+) - \hbar^2 + 2\hbar J_z\}|jm\rangle \\
&= J_-J^2|jm\rangle \\
&= j(j+1)\hbar^2 J_-|jm\rangle
\end{aligned}
$$

which implies that $J_-|jm\rangle = \alpha|j, m-1\rangle$, as it has the proper eigenvalues for J^2 and J_z. What is the normalization constant α? This can be determined only up to a phase (which is set to 1 here), by computing the norm of $J_-|jm\rangle$ (note that the adjoint state to $J_-|jm\rangle$ is $\langle jm|J_+$), with $J_-|jm\rangle = \sqrt{j(j+1) - m(m-1)}|j, m-1\rangle$, by the notes from Feb. 03 (class 07).

Computing Clebsch-Gordon coefficients

We add together two angular momenta (for example, spins of two particles, or the spin of a particle and its angular momentum). What we want to do is find the states with definite $J^{\rm tot}$ and definite $J_z^{\rm tot}$, where "tot" refers to the total angular momentum operators, with $(J^{\rm tot})^2 = (\vec{J}_1 I_2 + I_1\vec{J}_2)^2$ and $J_z^{\rm tot} = (J_{1,z}I_2 + I_1 J_{2,z})$, where the subscript refers to which angular momentum is being operated on. We can also define $J_\pm^{\rm tot} = (J_{1,\pm}I_2 + I_1 J_{2,\pm})$, where $J_{i,\pm} = J_{i,x} \pm iJ_{i,y}$.

There is a well defined procedure for constructing the $|jm\rangle$ states found by adding two angular momentum, using the product states $|j_1m_1\rangle|j_2m_2\rangle \equiv |j_1m_1; j_2m_2\rangle$ as a basis.

In brief, start with a state that has definite quantum numbers for total angular momentum and the z-component of angular momentum - express this state as a product state. Then apply total momentum lowering operators to this state to find the linear combinations of product states that give the other total J, J_z eigenstates.

1. START. Note that there is only one state with $J_z = j_1 + j_2$. That is the state $|j_1j_1; j_2j_2\rangle$. What is J^2? Well, one can write

   $$J^2 = J_1^2 + J_2^2 + 2J_{1,z}J_{2,z} + J_{1,+}J_{2,-} + J_{1,-}J_{2,+}\,;$$

   applying this operator to this state gives (note $J_{1,+}|j_1j_1\rangle = 0$, etc.)

   $$
   \begin{aligned}
   J^2|j_1j_1; j_2j_2\rangle &= [j_1(j_1+1) + j_2(j_2+1) + 2j_1j_2 + 0 + 0]|j_1j_1; j_2j_2\rangle \\
   &= (j_1+j_2)(j_1+j_2+1)|j_1j_1; j_2j_2\rangle\,,
   \end{aligned}
   $$

   so this state has $J = j_1 + j_2$. One can then conclude that

   $$|jm\rangle_{\rm tot} = |j_1j_1; j_2j_2\rangle\,.$$

2. APPLY J_- TO LOWER J_z. Given a state $|kk\rangle_{\rm tot}$ defined in terms of the product states, construct $|k, k-1\rangle$, $|k, k-2\rangle$, ..., $|k, -k\rangle$, by repeatedly applying $J_-^{\rm tot}$, using

   $$J_-|k, r\rangle = \sqrt{k(k+1) - r(r-1)}|k, r-1\rangle$$

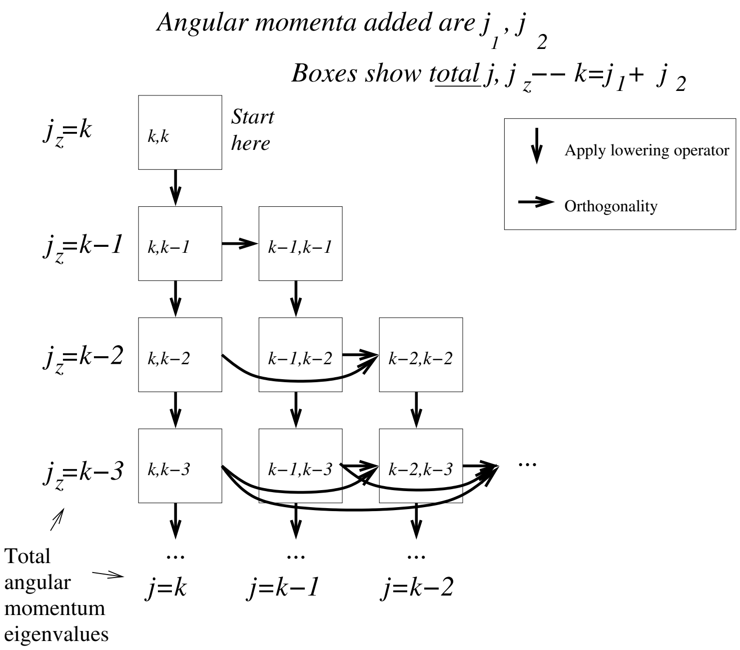
on one side, as this is for the total angular momentum representation, and

$$J_-|k_1, r_1; k_2, r_2\rangle = \sqrt{k_1(k_1+1) - r_1(r_1-1)}|k_1, r_1 - 1; k_2, r_2\rangle + \sqrt{k_2(k_2+1) - r_2(r_2-1)}|k_1, r_1; k_2, r_2 - 1\rangle$$

on the other side, in the product representation, since $J_- = J_{1,-}I_2 + I_1 J_{2,-}$.

3. USE ORTHOGONALITY TO DECREASE J. Once you have all of the states for $j = j_1 + j_2$, $j = j_1 + j_2 - 1$, ..., $j = j_1 + j_2 - p$, for example, you can construct the state $|j_1+j_2-p-1, j_1+j_2-p-1\rangle$ by finding the state with total $J_z = j_1+j_2-p-1$ that is orthogonal to all of the states $|j_1+j_2, j_1+j_2-p-1\rangle$, $|j_1+j_2-1, j_1+j_2-p-1\rangle$, ..., $|j_1+j_2-p, j_1+j_2-p-1\rangle$.

You can make a table to organize your calculations:



Note that there is one way to get $J_z = j_1 + j_2$, $|j_1 j_1; j_2 j_2\rangle$, but there are two product states with $J_z = j_1 + j_2 - 1$, i.e., $|j_1, j_1 - 1; j_2 j_2\rangle$ and $|j_1 j_1; j_2, j_2 - 1\rangle$. So linear combinations of these two product states will give $|j, j-1\rangle$ and $|j-1, j-1\rangle$.

In general, there are $(2j_1+1)(2j_2+1)$ independent states in the product basis. This procedure gives the $2(j_1+j_2)+1$ states with $j = j_1 + j_2$, the $2(j_1+j_2)-1$ states with $j = j_1 + j_2 - 1$, etc. If you add up this count through $j^{\text{tot,min}} = |j_1 - j_2|$, taking $j_1 \geq j_2$ for convenience, you get

$$\sum_{j=|j_1-j_2|}^{j=j_1+j_2} (2j+1) = 2 \sum_{j=|j_1-j_2|}^{j=j_1+j_2} j + \sum_{j=|j_1-j_2|}^{j=j_1+j_2} 1$$

$$
\begin{aligned}
&= 2\sum_{j=0}^{j=j_1+j_2} j - 2\sum_{j=0}^{j=|j_1-j_2|-1} j + j_1 + j_2 - j_1 + j_2 + 1 \\
&= (j_1+j_2+1)(j_1+j_2) - (j_1-j_2)(j_1-j_2-1) + 2j_2 + 1 \\
&= j_1^2 + j_2^2 + 2j_1j_2 + j_1 + j_2 - j_1^2 - j_2^2 + 2j_1j_2 + j_1 - j_2 + 2j_2 + 1 \\
&= 4j_1j_2 + 2j_1 + 2j_2 + 1 \\
&= (2j_1+1)(2j_2+1)\,.
\end{aligned}
$$

This shows that only the states with $j = |j_1 - j_2|, \ldots, j_1 + j_2$ show up in the total angular momentum representation. This is an important result.

Part of a sample calculation

Here $j_1 = 2$ and $j_2 = 1$.

First, $|33\rangle = |22\rangle|11\rangle = |22;11\rangle$.

Then apply lowering operators:

$$J_-|33\rangle = \sqrt{12-6}|32\rangle = (J_{1,-} + J_{2,-})|22;11\rangle = \sqrt{6-2}|21;11\rangle + \sqrt{2-0}|22;10\rangle\,,$$

so

$$|32\rangle = \frac{2}{\sqrt{6}}|21;11\rangle + \sqrt{\frac{1}{3}}|22;10\rangle\,.$$

Applying J_- again to $|32\rangle$ gives

$$\sqrt{12-2}|31\rangle = \frac{2}{\sqrt{6}}(\sqrt{6}|20;11\rangle + \sqrt{2}|21;10\rangle) + \sqrt{\frac{1}{3}}(\sqrt{6-2}|21;10\rangle + \sqrt{2}|22;1,-1\rangle)$$

or

$$|31\rangle = \sqrt{\frac{6}{15}}|20;11\rangle + \sqrt{\frac{8}{15}}|21;10\rangle) + \sqrt{\frac{1}{15}}|22;1,-1\rangle\,.$$

What does this mean? It means, for example, that if I know the total $J = 3$ and the total $J_z = 1$, then the probability that the first particle has $m_1 = 1$ is $\frac{8}{15}$. One could continue, but you don't need to if you are interested in the higher J_z states.

For example, $|22\rangle$ must be orthogonal to $|32\rangle$ yet have $J_z = 2$. The only way to get $J_z = 2$ is by combining $|21;11\rangle$and $|22;10\rangle$. By inspection, it follows that

$$|22\rangle = \sqrt{\frac{1}{3}}|21;11\rangle - \sqrt{\frac{2}{3}}|22;10\rangle$$

(up to a constant phase factor which is naturally chosen here). The other $J = 2$ states can then be found by applying J_- to $|22\rangle$.

In summary, this calculation was done in the following steps, where for each ket in the total angular momentum representation, we compute its representation as a sum of states in the product of the two component representations:

$$\begin{array}{lll}
|33\rangle & \Leftarrow (\mathrm{Step 1, Start}) & \\
\Downarrow (2,\ \text{using } J_-) & & \\
|32\rangle & \Rightarrow (4, \mathrm{orthogonality}) & |22\rangle \\
\Downarrow (3,\ \text{using } J_-) & & \\
|31\rangle & &
\end{array}$$