

PHY662, Spring 2004  
Outline for Tues. Jan. 27, 2004  
Spins in static and dynamic magnetic fields:  
precession and resonance

26th January 2004

## 1 Administration

### Today

1. Homework review: also a chance to review precession. Note  $\mu$ -decay: using  $g - 2$  to look for supersymmetry.
2. Conservation and symmetry.
3. Magnetic resonance.

## 2 Homework review

[See KEY for HWK #2 - we will spend more time on problem #3].

One of the more interesting scientific uses for precession is to look at  $g - 2$ . This is not zero, as predicted by, say, Dirac theory, due to corrections from, if you like, from virtual particles. The brief existence of extra particles affects the response of the particle to external magnetic fields.

For the electron,  $g = 2.002\,319\,304\,374 \pm 0.000\,000\,000\,008$ . This is extremely accurate and in good agreement with theoretical calculations. [Source: Particle Data Group, 2003.]

For the muon,  $g = 2.002\,331\,841\,6 \pm 0.000\,000\,001\,2$ . [From the BNL experiment, January, 2004; see [arxiv.org/hep-ex/0401008](http://arxiv.org/hep-ex/0401008).]

The experimental result for the muon is in *mild disagreement* with the theoretical predictions (1 or 2 standard deviations). Does this indicate the existence of unknown particles, such as supersymmetry? We will have to wait and see. The theoretical predictions are somewhat unsettled, actually. Also, disagreement at the several standard

deviation level often vanishes as more data is taken. But no one is taking data right now.

### 3 Conservation and Symmetry

We didn't have time last meeting to cover this properly. Here are the details, which I will review in class:

- A **symmetry** is a (time-independent) unitary operator that does not change the equations of motion.
- Let  $U$  be a symmetry operator. Then if  $|\psi\rangle$  obeys the equation of motion, so does  $U|\psi\rangle$ :

$$HU|\psi\rangle = i\hbar \frac{\partial}{\partial t} U|\psi\rangle.$$

This implies that

$$HU|\psi\rangle = i\hbar U \frac{\partial}{\partial t} |\psi\rangle = UH|\psi\rangle,$$

as  $|\psi(t)\rangle$  obeys the Schrodinger equation. Thus  $HU = UH$ . As  $U^\dagger U = 1$  for unitary operators, this can be written as

$$U^\dagger HU = I = UHU^\dagger.$$

- Suppose there is a set of symmetries that are parametrized by a single continuous parameter:  $U(a)$ . By composition,  $U(a)U(a) = U(2a)$ . Examples are translations by a distance  $a$  or rotations by an angle  $a$  about a given axis. Write  $U(a) = \exp(aG/i\hbar)$ . The operator  $G$  is said to be a *generator* (or sometimes charge) for the family of symmetries  $U(a)$ . Note that  $\frac{\partial}{\partial a} U(a) = \frac{G}{i\hbar} U$  and  $\frac{\partial}{\partial a} U^\dagger = \frac{\partial}{\partial a} e^{aG^\dagger/(-i\hbar)} = \frac{\partial}{\partial a} e^{-aG/i\hbar} = \frac{-G}{i\hbar} U$ . Since  $U(a)$  is unitary,  $G$  must be Hermitian. One way to see this is (using  $[G, U] = 0$ ) by noting

$$0 = \frac{\partial}{\partial a} U^\dagger U = \frac{G^\dagger}{-i\hbar} U^\dagger U + U^\dagger \frac{G}{i\hbar} U = -G^\dagger + G.$$

We can use the same trick, this time taking the derivative of  $H$ , rather than 1,

$$0 = \frac{\partial}{\partial a} H = \frac{\partial}{\partial a} (U^\dagger HU) = \left(\frac{G^\dagger}{-i\hbar}\right) U^\dagger HU + U^\dagger H \left(\frac{G}{i\hbar}\right) U.$$

This gives

$$GH - HG = [G, H] = 0.$$

- Applying Ehrenfest's theorem to  $G$  then gives  $\frac{d}{dt} \langle G \rangle = \langle [G, H] \rangle = 0$  whenever  $G$  is a generator of a symmetry  $U(a)$  for a given Hamiltonian.

## 4 Comments

The study of spin-1/2 particles is fundamental, as everyday matter is made of spin-1/2 particles. The technology for spin-1/2 systems is also used in the study of *two-level systems*, which are the simplest non-trivial systems. Often, only two states of a system are important, so knowing the behavior of two-level systems is very useful.

## 5 Spins in a constant magnetic field

To repeat from last time, the calculation of the effect of a magnetic field on a spin-1/2 particle is straightforward. We take the Hamiltonian for a spin in a magnetic field  $\vec{B}$  to be

$$\mathcal{H} = -\vec{\mu} \cdot \vec{B} = -\gamma \vec{S} \cdot \vec{B}.$$

The Schrodinger equation  $i\hbar \frac{\partial}{\partial t} |\psi\rangle = \mathcal{H} |\psi\rangle$  has as a solution (when  $\mathcal{H}$  is independent of time)

$$|\psi(t)\rangle = e^{\mathcal{H}t/i\hbar} |\psi(t=0)\rangle.$$

This immediately gives

$$\begin{aligned} |\psi(t)\rangle &= e^{-i\mathcal{H}t/\hbar} |\psi(0)\rangle \\ &= e^{-i(-\gamma t \vec{B}) \cdot \vec{S}/\hbar} |\psi(0)\rangle \\ &= e^{-i\vec{\theta} \cdot \vec{S}/\hbar} |\psi(0)\rangle, \end{aligned}$$

which is just the operator that rotates the state by an angle of  $\vec{\theta} = -\gamma \vec{B}t$ . This means that all expectation values, such as the vector  $\langle \vec{S} \rangle$ , can be found by rotation by an amount proportional to time. So we speak of the spin “precessing”, as the expectation value itself “rotates” at angular frequency  $\gamma |\vec{B}|$ .

## 6 Magnetic resonance

Here, we will follow the outline of the calculation given in Baym.

The key result is this:

If you put a spin in a constant magnetic field, there is a well-defined precession rate  $\omega_0 = \gamma |\vec{B}_0|$ . If you in addition apply an oscillatory magnetic field of strength  $B_1$  of frequency  $\omega$  that is near to  $\omega_0$ , the up/down amplitudes will strongly oscillate. The frequency of this oscillation in the amplitudes is  $\omega_1 = \gamma B_1/2$ . The amplitude is maximal when  $\omega = \omega_0$  (resonance). As the energy of the spin depends on its orientation, this change in amplitude means that the spin is absorbing or radiating electromagnetic energy.

The calculation we will follow will be slightly simplified compared with Baym's discussion by a slightly different (impractical) choice for the oscillatory external magnetic field. The choice will be  $\vec{B}_1 = \frac{B_1}{2} \cos(\omega t) \hat{x} - \frac{B_1}{2} \sin(\omega t) \hat{y}$  (the more realistic calculation uses only one of these components, i.e.,  $\vec{B}_1$  is linearly polarized in real life; the factor of  $\frac{1}{2}$  makes the results agree with the linear polarized case  $\vec{B}_1 = B_1 \cos(\omega t)$ ). Let the static magnetic field be given by  $\vec{B}_0 = B_0 \hat{z}$ . The spin Hamiltonian is then

$$\mathcal{H} = -\gamma \vec{S} \cdot \vec{B} = -\gamma [B_0 S_z + \frac{B_1}{2} S_x \cos(\omega t) - \frac{B_1}{2} S_y \sin(\omega t)].$$

The primary time dependence comes from  $B_0$  and we are working in the  $\hat{z}$ -basis, so it is reasonable to remove the time independence of the precession in the static field by rewriting the wave function (spinor)  $|\psi\rangle$  as a product of the primary time dependence and a "slow" part,  $|\psi'(t)\rangle$ :

$$|\psi(t)\rangle = e^{i\omega t \sigma_z / 2} |\psi'(t)\rangle.$$

The Schrodinger equation  $i\hbar \frac{\partial}{\partial t} |\psi\rangle = \mathcal{H} |\psi\rangle$  is then (since  $S_z = \frac{\hbar}{2} \sigma_z$ , etc. and setting  $\omega_1 = \gamma B_1 / 2$ ):

$$\begin{aligned} i\hbar \frac{\partial}{\partial t} e^{i\omega t \sigma_z / 2} |\psi'(t)\rangle &= [-\gamma B_0 S_z - \gamma \frac{B_1}{2} S_x \cos(\omega t) + \gamma \frac{B_1}{2} S_y \sin(\omega t)] e^{i\omega t \sigma_z / 2} |\psi'(t)\rangle \\ &= \hbar [-\frac{1}{2} \omega_0 \sigma_z - \frac{\omega_1}{2} \sigma_x \cos(\omega t) + \frac{\omega_1}{2} \sigma_y \sin(\omega t)] e^{i\omega t \sigma_z / 2} |\psi'(t)\rangle \end{aligned}$$

Since  $\frac{\partial}{\partial t} e^{i\omega t \sigma_z / 2} |\psi'\rangle = \frac{i}{2} \omega \sigma_z e^{i\omega t \sigma_z / 2} |\psi'\rangle > + e^{i\omega t \sigma_z / 2} \frac{\partial}{\partial t} |\psi'\rangle >$ , multiplying both sides by  $e^{-i\omega t \sigma_z}$  and cancelling out the  $\hbar$ 's gives the equation

$$-\frac{\omega}{2} \sigma_z |\psi'\rangle + i \frac{\partial}{\partial t} |\psi'\rangle = -\frac{1}{2} \omega_0 \sigma_z |\psi'\rangle + e^{-i\omega t \sigma_z / 2} [\frac{\omega_1}{2} \sigma_x \cos(\omega t) - \frac{\omega_1}{2} \sigma_y \sin(\omega t)] e^{i\omega t \sigma_z / 2} |\psi'(t)\rangle.$$

Now,  $\sigma_z$  anticommutes with  $\sigma_x$  and  $\sigma_y$ . So  $\sigma_z \sigma_x = -\sigma_x \sigma_z$ . But

$$\sigma_z^2 \sigma_x = -\sigma_z \sigma_x \sigma_z = \sigma_x \sigma_z^2.$$

This means that odd powers of  $\sigma_z$  anticommute with  $\sigma_x$  while even powers commute. A similar demonstration holds for  $\sigma_y$ . Using the expansion for the exponential, this means that

$$e^{-i\omega t \sigma_z / 2} \sigma_x = \sigma_x e^{i\omega t \sigma_z / 2},$$

giving

$$\begin{aligned} i \frac{\partial}{\partial t} |\psi'\rangle &= \left\{ \frac{\omega - \omega_0}{2} \sigma_z - \frac{\omega_1}{2} [\sigma_x \cos(\omega t) - \sigma_y \sin(\omega t)] e^{i\omega t \sigma_z} \right\} |\psi'\rangle \\ &= \left\{ \frac{\omega - \omega_0}{2} \sigma_z - \frac{\omega_1}{2} [\sigma_x \cos(\omega t) - \sigma_y \sin(\omega t)] [\cos(\omega t) + i \sin(\omega t) \sigma_z] \right\} |\psi'\rangle \\ &= \left\{ \frac{\omega - \omega_0}{2} \sigma_z - \frac{\omega_1}{2} \begin{bmatrix} 0 & \cos(\omega t) + i \sin(\omega t) \\ \cos(\omega t) - i \sin(\omega t) & 0 \end{bmatrix} \begin{bmatrix} e^{i\omega t} & 0 \\ 0 & e^{-i\omega t} \end{bmatrix} \right\} |\psi'\rangle \end{aligned}$$

$$\begin{aligned}
&= \left\{ \frac{\omega - \omega_0}{2} \sigma_z - \frac{\omega_1}{2} \begin{bmatrix} 0 & e^{i\omega t} \\ e^{-i\omega t} & 0 \end{bmatrix} \begin{bmatrix} e^{i\omega t} & 0 \\ 0 & e^{-i\omega t} \end{bmatrix} \right\} |\psi'\rangle \\
&= \left\{ \frac{\omega - \omega_0}{2} \sigma_z - \frac{\omega_1}{2} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\} |\psi'\rangle \\
&= \left( \frac{\omega - \omega_0}{2} \sigma_z - \frac{\omega_1}{2} \sigma_x \right) |\psi'\rangle.
\end{aligned}$$

As the operator  $\Omega = (\omega - \omega_0)\sigma_z - \omega_1\sigma_x$  is independent of time, the solution to this equation for  $|\psi'\rangle$  is

$$\begin{aligned}
|\psi'(t)\rangle &= e^{-it\hat{\Omega}/2} |\psi'(0)\rangle \\
&= e^{-it\Omega t\hat{\Omega}/2} |\psi'(0)\rangle,
\end{aligned}$$

where  $\Omega = [(\omega - \omega_0)^2 + \omega_1^2]^{1/2}$  and  $\hat{\Omega} = \vec{\Omega}/\Omega$ . Substituting back in for the real spinor gives

$$|\psi(t)\rangle = e^{i\omega t\sigma_z/2} e^{-i\Omega t\hat{\Omega}/2} |\psi(0)\rangle.$$

This expression can be used to calculate the amplitudes for the spinor, given that the particle is in a static magnetic field and a small oscillating one.

For example, suppose  $|\psi(0)\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$  and  $\omega = \omega_0$ . Then  $\hat{\Omega} = -\omega_1\sigma_x$  is just proportional to the “spin-flip” operator  $\sigma_x = \frac{1}{\hbar}(S_+ + S_-)$ . Up spins are turned down and down spins are turned up. The operator  $e^{-\Omega t\hat{\Omega}/2}$  just rotates the spinor by an angle  $\Omega t = \omega_1 t$  in this case. This leads to an oscillation in the populations of the up and down spins, with the probability switching between the two states becoming unity at time  $\pi\omega_1^{-1}$ . If the oscillating field has this duration, it is called a “ $\pi$ -pulse”. (The  $e^{i\omega t\sigma_z/2}$  in front gives a relative phase to the up/down amplitudes, *after* this spin flip has been applied and can be ignored for calculating the relative probabilities of up/down, but not other probabilities.)