PHY662, Spring 2004
Outline for Thurs. Jan. 15, 2004

15th January 2004

Administration

- Hand out homework: see for reading and problems.
- Future assignments - will see about handing out problem sets on Tuesday.

Aside: deforming sequences of rotation - not required reading

Here is a slightly more formal discussion of paths in rotation space: this just reinforces what we went over for (1) the twisted bands and (2) the composition of rotations of a sphere. It is for those who might be worried about the mathematical niceties of the discussion in the first lecture.

The elements of $SO(3)$, the group of proper rotations (i.e., rotations that do not contain reflections) for a 3D space, form a manifold that has three dimensions. A continuous sequence of rotations is a map from the interval $[0, 1]$ to $SO(3)$, $t \rightarrow R$, that is continuous [if δ is small, $R(t)$ is nearly the same as $R(t+\delta)$]. Let $R(t)$ be a sequence of rotations with $R(t) = R(4\pi t\hat{z})$, where $R(\vec{\theta})$ is a rotation by an angle θ about the axis $\hat{\theta}$. Then $R(0) = R(1/2) = R(1) = I$, where I is of course the identity (no rotation). This continous sequence can itself be continously deformed to the trivial sequence $R(t) = I$. That is, one can define a continuous map $S(u, t)$ from $[0, 1] \times [0, 1] \rightarrow SO(3)$ with $S(0, t) = R(t)$ and $S(1, t) = I$. We take two 2π twists and convert it to the identity, continously. The idea is to transform the second 2π twist to a 2π twist ***opposite*** the first 2π twist and then to cancel out the two twists. This was done pictorially in the first lecture. More formally, this can be done by

- setting, for $0 \leq u \leq \frac{1}{2}$,
  - $S(u, t) = S(0, t) = R(4\pi t\hat{z})$ when $0 \leq t \leq \frac{1}{2}$,
  - $S(u, t) = R[4\pi t\hat{n}(u)]$ for $\frac{1}{2} \leq t \leq 1$, where $\hat{n}(0) = \hat{z}$ continously changes to $\hat{n}(\frac{1}{2}) = -\hat{z}$, and then

- Now at $u = \frac{1}{2}$, write the sequence of rotations as a function of t as $R(\theta(t)\hat{z})$, with $\theta(t)$ increasing from 0 at $t = 0$ to $\theta = 2\pi$ at $t = \frac{1}{2}$ and then decreasing back to $\theta = 0$ at $t = 1$, using $R(\theta(-\hat{z})) = R((2\pi - \theta)\hat{z})$. You can then cancel out the sequence of rotations over the range $\frac{1}{2} \leq u \leq 1$ (reducing the rotations by cancelling them out).

The import of this result is that you might be able to get away with having results depend on 2π rotations (or the exchange of 2 fermions!) but a sequence of rotations resulting in 4π is continously linked to doing nothing.

Discussion of Feynman's Derivation

Review Feynman, et al., Sections 6-1 through 6-4 (reading was 6-1 through 6-3).

Assumptions:

[Some of these assumptions are to some extent derived in Ch. 5 of Feynman, et al., or at least their consistency explained.]

- The description of the state of a particle can be described by the amplitudes, complex numbers, over some set of base states.
- The states $|i\rangle$ in the basis are complete, with the projection operators satisfying $\sum_i |i\rangle\langle i| = I$.
- The inner product between two states is skew-symmetric $\langle i|j\rangle = \langle j|i\rangle^*$.
- Note that Feynman uses "representation" of a transformation to mean the particular matrix corresponding to a rotation in a given basis (rather than the more abstract representation of a group by a set of matrices).

Preliminaries

- The S-G apparatus sorts atoms by their magnetic moment $\vec{\mu}$. As the most that rotations of the apparatus can do is select new directions in space, the transformation of states of the atoms is a representation of the *generators* of $SO(3)$. [It turns out that the group we will construct is $SU(2)$, which has the same local geometry as $SO(3)$ but has a distinct topology.]
- An "open" apparatus does not change $|\psi\rangle$.
- Blocking a beam removes the corresponding component of $|\psi\rangle$. For example, blocking the down beam sets the down amplitude to zero, while leaving the up amplitude unaffected.
- The coefficients of transformations depend only on relative rotations and the composition of transformations between base sets is given by the multiplication of matrices.

- Feynman shows that you can choose Det$R = |R| = 1$. [Note: gives $SU(2)$ as the most general set of unitary transformations with determinant 1, i.e., linear transformations that preserve the complex inner product and have determinant 1.]

z-rotations

- "Surely" up is maintained when the apparatus is rotated about the z-axis. What are the implications for the amplitudes in the rotated basis?
- Fix the relative phases between up and down for a given rotation (according to the condition on $|R|$, rather than the more complicated way that Feynman describes separately).
- Show that composition of two rotations about z gives linearity.
- "We must have the situation that a rotation by 360^o and *no smaller angle* reproduces the same physical state." (Same physical state does not mean same amplitude, though). This is based on the result of feeding in a beam "polarized" in the $\hat{x}$ direction into a sequence of 2 $\hat{z}$-oriented S-G apparatuses. What if there were more spin states??

y-rotations

- Rotation by π about $\hat{y}$ must give

$$C_+^{'} = e^{i\beta} C_- , \qquad C_-^{'} = e^{i\gamma} C_+ \ .$$

- 2π rotation about $\hat{y}$ must give the same phase change as a 2π rotation about $\hat{z}$. This gives a constraint on γ and β. Then set γ by convention to get Eq. (6.22).

Following Shankar, by using $[S_i, S_j] = i\hbar\epsilon_{ijk}S_k$ and assuming 2 states

We will use implicit summations: $a_i b_i = \sum_i a_i b_i$.

Note: Shankar writes vectors as **A**. I will use $\vec{A}$.

In earlier chapters, when studying orbital angular momentum, the commutation relations $[J_i, J_j] = i\hbar\epsilon_{ijk}J_k$ are used to determine the finite-dimensional representations for the angular momentum operators $\vec{J}$. This is done especially through the use of the raising and lowering operators $J_+ = J_x + iJ_y$ and $J_- = L_x - iL_y$ and commutation relations between them and J_z and $J^2 = J_x^2 + J_y^2 + J_z^2$, with $J^2 = J_- J_+ + J_z^2 - \hbar J_z$. Uniqueness of the wave function in real space (x, y, z points) implies orbital angular momenta are quantized in units of $\hbar$.

Discussion of important expressions:

$$[S_i, S_j] = i\hbar\epsilon_{ijk}S_k$$

has as a representation

$$\vec{S} = \frac{\hbar}{2}\vec{\sigma}$$

where the Pauli matrices are defined as

$$\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

The σ_i anticommute, with

$$\sigma_i\sigma_j = -\sigma_j\sigma_i \quad (i \neq j)$$

and are square roots of the identity:

$$\sigma_i^2 = I$$

and are traceless.

Show that $(\hat{n} \cdot \sigma)^2 \vec{=} I$.

Establish the expression

$$(\vec{A} \cdot \vec{\sigma})(\vec{B} \cdot \vec{\sigma}) = \vec{A} \cdot \vec{B} + i(\vec{A} \times \vec{B}) \cdot \vec{\sigma},$$

for vector operators that commute with $\vec{\sigma}$.

In the $\hat{n}$ basis, the base states are $|\hat{n}, +\rangle$ and $|\hat{n}, -\rangle$, are eigenvectors of the operator $\hat{n} \cdot \vec{S}$, with the same eigenvalues for $\vec{S}$ of $\pm\hbar/2$,

Next time, we will use the fact that 2×2 matrices can be represented as linear combinations of Pauli matrices.

Carry out derivation of 14.3.39, after introducing commutation and anticommutation relations for the σ_i.