PHY662 - Quantum Mechanics II
HWK #10, Due Thursday, Apr. 8, *start* of class

- Reading:
  - Read up to (but not including) the "higher orders" (Sec. 18.3) in Shankar and start to read Sec. 18.4 on electromagnetism.
  - Sections 9.1 and 10.1 of Griffiths might be useful for review and a different perspective.

1. *Fermi's golden rule applied in $d = 1$.* [3 pts] Consider a particle free to travel in one dimension, along the x-coordinate.
   (a) To keep track of normalization, it is often useful to put the free particle in a box of dimension L and take the limit as L becomes large. For boundary conditions, it is easiest to use *periodic boundary conditions* with $\psi(-L/2) = \psi(L/2)$. Given these boundary conditions, what are the eigenstates for this particle?

   (b) Let the particle initially be at rest. Let a potential defined over all space of the form $V(x) = u\cos(kx - \omega t)$ be turned on. Use Fermi's golden rule to calculate the total rate that the particle will be excited to a moving state (what are the possible final moving states or state, to first order?), in the limit of a truly free particle, that is, $L \to \infty$.

2. *Transition rates in a 3D case.* [5 pts] Consider an α-particle of charge $2e$ in a three-dimensional box of sides L, with L large enough that that the eigenstates can be considered to form a continuous spectrum. The position vector coordinates are in the range $-L/2 < x, y, z \leq L/2$. Use periodic boundary conditions, $\psi(\vec{r} + jL\hat{x} + kL\hat{y} + mL\hat{z}) = \psi(\vec{r})$, for integer j, k, m. You will consider the effect of a perturbing potential, the Coulomb potential centered at the origin, on an electron in this box.
   (a) What are the eigenstates for a free particle (the α-particle) in the periodic box with sides L?

   (b) What is the Fourier transform of the Coulomb potential felt by a charge $2e$ that is caused by a fixed charge of value Ze? That is, what is the Fourier transform of $\frac{2Ze^2}{r}$?
      i. Everyone should do this integral sometime. One convenient trick is to introduce an exponential cutoff into the Coulomb potential, then remove it by taking a limit. An exponential cutoff of the Coulomb potential gives the Yukawa potential $\frac{(2Ze^2)e^{-\kappa r}}{r}$ (note the terrible notation: there are two distinct e's here. The parameter κ sets the range of the exponential cutoff.

   ii. To find the Fourier transform of the pure Coulomb potential, you want to find the Fourier transform in the limit $\kappa \to 0$, $\tilde{V}(\vec{k}) = \lim_{\kappa\to 0} \int d^3r\, e^{i\vec{k}\cdot\vec{r}}(\frac{(2Ze^2)e^{-\kappa r}}{r})$.

   iii. The result is independent of the direction of $\vec{k}$, so pick $\vec{k} = k\hat{z}$. Of course, work in spherical coordinates, so you want to compute the integral $\int_0^\infty dr \int_0^\pi \sin(\theta) d\theta \int_0^{2\pi} d\phi \frac{(2Ze^2)e^{ikr\cos\theta - \kappa r}}{r}$. Change variables $y = \cos(\theta)$ and do the integral over y first.

   iv. After integrating over r, take $\kappa \to 0$ to get the answer.

(c) Let an α-particle of charge $2e$ be in motion with momentum $\vec{p} = \hbar\vec{k}$, in a box of size L. Let there be a very massive (fixed) nucleus of charge Ze at the origin. What is the rate for the particle to make a transition to another state with a given momentum $\vec{p}' = \hbar\vec{k}'$? Use Fermi's golden rule with $\omega = 0$ and express your answer using a δ-function. NOTE: your answer for the transition rate will depend on the box size.

(d) Suppose the α-particle of charge $2e$ is moving at the speed 100 km/s along the x-axis, in a periodic box of size $L^3 = 1\,\mathrm{cm}^3$ containing a nucleus of charge Ze at its center. (This corresponds to an electron traveling in infinite space with protons present at a density of 1 cm^{-3}.) The initial speed defines your initial state $|i\rangle$ with wavector $\hbar\vec{k} = \vec{p} = m\vec{v}$. What is the rate, measured in Hz, at which the electron will make a transition to a state with a velocity nearly perpendicular to the original direction, where "nearly perpendicular" means that the new direction is perpendicular to the original direction to within about 0.001 radians? More precisely, your final state $|f\rangle$ has $|\vec{k}_f \cdot \vec{k}_i| < 0.001(\vec{k}_i)^2$. You may take $|\vec{k}_f - \vec{k}_i|$ to be constant over this range of final states. You will need to integrate over final states using the δ-function and the density of states $\rho(\vec{k}) = (2\pi/L)^3$.

3. *Gauge invariance.* [2 pts; This is just Shankar's exercise 18.4.4.]

   (a) Given potentials $\vec{A}$ and ϕ, write the Hamiltonian H.

   (b) Given a gauge field $\Lambda(\vec{r}, t)$, write down the Hamiltonian H_Λ for the transformed potentials $\vec{A}_\Lambda, \phi_\Lambda$.

   (c) Show that if $\psi(\vec{r}, t)$ is a solution to Schrodinger's equation with the Hamiltonian H, then $\psi_\Lambda(\vec{r}, t) = e^{-iq\Lambda(\vec{r},t)}$ is a solution to Schrodinger's equation with the Hamiltonian H_Λ.