PHY662 - Quantum Mechanics II
HWK #9, Due Tues., Mar. 30, at the *start* of class

- Reading:
  - Read up to (but not including) the "higher orders" (Sec. 18.3) in Shankar and start to read Sec. 18.4 on electromagnetism.
  - Sections 9.1 and 10.1 of Griffiths might be useful for review and a different perspective.

1. *Practice with creation and annihilation operators.* [2 pts] Consider a particle in a 1D harmonic well.

   (a) [This is just Ex. 18.2.1 in Shankar] Let your particle be in the ground state $|0\rangle$ at large negative times t. Expose the particle to the time-dependent perturbation $H^1(t) = -e\mathcal{E}X/[1 + (t/\tau)^2]$. This corresponds to an electric field of time varying strength. To first order in the time-dependent part, compute the probability that the particle will be in the first excited state of the well at large positive times t.

   (b) Let your particle be in the *third* excited state $|3\rangle$ at large negative time t. For the perturbation of part (a), what are the possible states the particle can be in at large positve t, if you compute transition probabilities to first order in the perturbation? What are the probabilities for the particle to be in these other states, to first order in $\mathcal{E}$?

2. *Different types of changes in the Hamiltonian.* [3 pts] Again, consider a particle in a 1D harmonic well, with Hamiltonian $H^- = \frac{k}{2}X^2 + \frac{1}{2m}P^2$ at large negative time t. Let the particle *initially* be in the *ground state*, that is $|\psi(t = -\infty)\rangle = |0\rangle$, in each part below.

   (a) If H is equal to H^- for all $t < 0$ and $H = H^+ = \frac{k'}{2}X^2 + \frac{1}{2m}P^2$ for $t > 0$, what is the probability that the particle is in the ground state of the new Hamiltonian H^+ at positive t?

   (b) If $H(t) = H^-$ for $t < 0$, $H(t) = (1 - \frac{t}{T})H^- + (\frac{t}{T})H^+$ for $0 < t < T$, $H(t) = H^+$ for $t > T$, what is the probability that the particle is in the ground state of H^+ at time T, in the limit of large T?

   (c) If $H(t) = H^- + uX^2\delta(t)$, what is the probability that the particle is in an excited state of H^- at large positive t? (Note that $H(t) =$Compute your answer using time-dependent perturbation theory to first order in u and the notes from class. (Please note that, to first order, there is only one excited state that can be reached by this perturbation - during your calculation, you should determine what state the particle can be excited to.)

3. *Lenz's Law via quantum mechanics.* [5 pts] Classically, when you apply a time-changing magnetic field to a loop of conducting material, you create a current in the loop that opposes the change in the magnetic field. This is Lenz's law. Based on last week's homework (#8), consider the quantum dynamics of a spinless electron in a circular ring. ("Spinless" means we neglect the spin degree of freedom, so that there is no $-\vec{\mu} \cdot \vec{B}$ part of the Hamiltonian). You will compute how a changing magnetic field might induce currents in a ring that confines a single electron.

First, for review of notation and concepts, consider the general problem of a charged particle in the presence of electromagnetic fields. The vector potential $\vec{A}$ is a field such that the magnetic field can be written as $\vec{B} = \nabla \times \vec{A}$. It is possible to write $\vec{B}$ in this fashion, using $\nabla \cdot \vec{B} = 0$ and vector calculus. The electric field is then $\vec{E} = -\nabla\phi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t}$, where ϕ is the electrostatic potential. Remember also that $\vec{A}$ can be transformed by adding a gradient of a scalar field, $\vec{A} \to \vec{A} + \nabla\chi(\vec{r})$, without affecting $\vec{B}$ or any other physically measurable quantity. This insensitivity is called gauge invariance. If $\vec{A} = 0$, the kinetic part of the Hamiltonian is $\frac{1}{2\mu}\vec{p}^2$ and the current of a particle is given by $\vec{j}(\vec{r}) = (\psi^*(\vec{r})\nabla\psi(\vec{r}) - \psi(\vec{r})\nabla\psi^*(\vec{r}))/2\mu$ (here "current" is not electric current - this is just the velocity times the density - to get the *electric* current, you need to multiply this particle current by the charge q). For the Hamiltonian with $\vec{A} \neq 0$, the kinetic part of the Hamiltonian is

$$\frac{1}{2\mu}\left(\vec{p} - \frac{q\vec{A}}{c}\right)^2 .$$

(When $\phi \neq 0$, the full Hamiltonian is $\frac{1}{2\mu}(\vec{p} - \frac{q\vec{A}}{c})^2 + q\phi$, but for this problem, we will take $\phi = 0$.) This part of the Hamiltonian can be derived from a correspondence with classical mechanics or can be viewed as the simplest gauge invariant Hamiltonian for a particle interacting with the electromagnetic field. In the presence of a vector potential $\vec{A}$, the locally conserved particle current is

$$\vec{j}(\vec{r}) = \frac{\hbar}{2i\mu}\left(\psi^*\nabla\psi - \psi\nabla\psi^*\right) - \frac{q}{\mu c}\psi^*\psi\vec{A} .$$

It is easily shown from the Hamiltonian in a presence of a magnetic field that

$$\frac{\partial\rho}{\partial t} = -\nabla \cdot \vec{j} ,$$

where the particle density $\rho(\vec{r}) = \psi^*(\vec{r})\psi(\vec{r})$. This justifies the definition of the current $\vec{j}(\vec{r})$. (You can try to derive this yourself, looking up discussions of this topic for hints. We will also review how to do this in class.)

(a) Recall the physical setup and key from HWK #8 (electron confined to a ring of radius R). For arbitrary values of magnetic field of strength B perpendicular to the ring, compute the expectation value of the particle current $\vec{j}$ for the exact eigenstates $|m\rangle$of the Hamiltonian for a *spinless* electron. (The currents, of course, must be in the $\hat{\theta}$ direction - you are to find the strength and direction of these currents for a given value of m). Note that any gradients of ψ or ψ^* are in the $\hat{\theta}$ direction.

(b) Suppose the electron is initially in the ground state for $B = 0$. If you *very quickly* turn up the external magnetic field to a final value B, what is the expectation value of the current around the ring after the field is raised?

(c) Suppose the electron is initially in the ground state for $B = 0$. If you *very slowly* turn up the external magnetic field to a final value B, what is the expectation value of the current around the ring after you are done raising the field?

(d) In SI units, the flux quantum (which is hc/e in Gaussian units) is $hc/e = 4.14 \times 10^{-15}\,\mathrm{T} \cdot \mathrm{m}^2$. For reference, note that 10 T is a strong field in experiments, 1 T is not too hard to achieve, and that the Earth's magnetic field is of the order of $50\,\mu\mathrm{T}$. If you had a ring of radius $1\,\mu\mathrm{m}$ containing an electron in its ground state at $B = 0$ and suddenly turned up the field to $0.01\,\mathrm{T}$ ($= 100\,\mathrm{G}$), what would the velocity of the electron be after raising the field?