PHY662 - Quantum Mechanics II
HWK #6, Due Tues., Feb. 24, at the *start* of class

- Reading: Read Ch. 8 of Griffiths, *Introduction to Quantum Mechanics.*

1. *Airy functions.* Consider the ψ and Υ mesons. They are composed of quark-antiquark ($q\overline{q}$) pairs, respectively, a $c\overline{c}$ and $b\overline{b}$ pair, where b and c indicate the bottom and charm quarks. The quarks are held together by the strong force. The $q\overline{q}$ pairs can be in different orbital states, leading to different excitations above the ground state. Different excitations of these pairs appear as different particles in accelerator experiments.
   This problem will explore estimates for the masses of these mesons. The masses will be given by the energies of the excited states. Masses will be measured in units of GeV/c^2. We will only study the spectrum of the S ($\ell = 0$) states. The excitations are only in the radial wavefunction, i.e., a function of the relative separation of the quarks. The masses of the $c, \overline{c}$ quarks are m_c =1.81 GeV/c^2 each. The masses of the $b, \overline{b}$ quarks are m_b =5.25 GeV/c^2 each. $q\overline{q}$ mesons can have masses less than $2m_q$, as they have negative potential energy, being bound states.

   (a) *Setting up.* The potential between a quark q and an antiquark $\overline{q}$ is often taken to be linear, $V(r) = V_0 + Kr$. (This is an approximation to the character of the strong force - note that it takes an infinite amount of energy to separate a pair of quarks.) Write down the eigenvalue problem for the time-independent Schrodinger equation of two particles of mass m with radial separation coordinate r, using this potential. As we are setting $l = 0$, this is a single coordinate (radial) equation. It would be best to write the wave equation for $U(r) = r\psi(r)$, as is usually done for the hydrogen atom.

   (b) *Boundary conditions.* What are reasonable boundary conditions for $r\psi(r)$? ***Justify your answer.***

   (c) *Solving the eigenvalue problem.* Given the Schrodinger equation and boundary conditions, find the 4 lowest eigenenergies for the two $q\overline{q}$ bar states, in terms of V_0 and K. Solve the Schrodinger equation exactly: do not use the WKB approximation as Shankar does. [Hint #1: carry out a change of coordinates to change the Schrodinger equation to the differential equation which has the Airy functions as solutions.] To find the eigenenergies, you will need some information about the Airy function. [Hint #2: go to Google, enter "airy" and "nist".]

   (d) *Check with reality.* How well can you fit experimental results? If the mass of the 1S and 2S states of the ψ particle are 3.096 GeV/c^2 and 3.685 GeV/c^2, can you compute V_0 and K? Given these values, what would you predict for the mass of the 3S state for ψ?

(e) The 1S and 2S $\Upsilon = b\bar{b}$ states have masses 9.460 GeV/c^2 and 10.023 GeV/c^2. What is V_0 and K in this case? What would you predict for the mass of the 3S state for Υ? Compare with the value of $m(\Upsilon(3S))$ =10.355 GeV/c^2. What about $m(\Upsilon(4S))$=10.580 GeV/c^2 — does this fit well into the spectrum?

(f) *Relativity?* How fast are the quarks moving in these sets of mesons, roughly, compared with the speed of light? Which set of mesons (ψ or Υ) should be better described by your calculations?

(g) BONUS. (I haven't worked this all out yet to check if it is doable - so beware. I think it should work, though, with some numerical effort.) If you have extra time, fix up the energy estimates by trying a different model. At close distances the quarks attract. This is often described by a Coulombic attraction. Here, try something simpler: add a δ-function attraction $-u\delta(r)$ to the potential. By varying u, how well can you match the spectrum 9.460, 10.023, 10.355, 10.580 GeV/c^2 for the $n = 1, 2, 3, 4$ S states of $b\bar{b}$?

2. *Using WKB with the simplest connection formulas.* In this problem, you will use the WKB approximation to compute the spectrum of eigenstates (set of allowed energy values) for a particle in a potential of the form

$$V(x) = \begin{cases} \infty & x < -a \\ F|x| & -a < x < a \\ \infty & a < x \end{cases} .$$

Assume that there there are no classical turning points for $-a < x < a$. That is, assume that the energy eigenvalue E satisfies $E > V(x)$ for all $-a < x < a$.

(a) Write down the equation you would need to solve to find the eigenvalues E_n for a given F.

(b) Check that your expression gives the correct eigenvalues in the limit $F \to 0$.

(c) Solve this equation numerically to compute the ground state energy E_0 and the first excited state E_1, to 6 significant figures (within this approximation), when $F = \kappa\frac{\hbar^2\pi^2}{8ma^3}$, for the two cases $\kappa = 0.2$ and $\kappa = 0.5$. That is, compute $E_0(\kappa = 0.2)$, $E_1(\kappa = 0.2)$, $E_0(\kappa = 0.5)$, and $E_1(\kappa = 0.5)$. [Hint: to reduce errors, scale the equation by substituting $E = \lambda\frac{\hbar^2\pi^2}{8ma^2}$ and solve for λ. State your answers by giving values for λ_1 and λ_2]