PHY662 - Quantum Mechanics II
HWK #4, Due Tues., Feb. 10, at the *start* of class

- Reading: Read through p. 417 of Shankar by Thursday, Feb. 5, and then through Sec. 15.3 (top of p. 421) by Tuesday, Feb. 10.

1. *Resolution estimate for MRI. In practice, many factors affect the resolution of MRI images. Here you will consider one theoretical limit. You will compute how thin one can slice a sample.*
   To excite the spins in a thin 2D slice centered at $z = z_s$, one first adds a z-gradient to the static field B_0, $\vec{B}_0 = [B_0 + B_g z]\hat{z}$ and, while keeping the gradient on, applies a $\frac{\pi}{2}$-pulse intended for protons precessing at frequency $\omega = \gamma[B_0 + B_g z_s]$. (You would then analyze the signal using phase and frequency encoding, but that is not addressed in this problem.) Let T be the duration of the $\frac{\pi}{2}$-pulse. During this pulse, $\vec{B} = \vec{B}_0 + \vec{B}_1$, with $\vec{B}_1 = B_1 \cos(\omega t)\hat{x}$ for $0 \le t \le T$, otherwise $\vec{B}_1 = 0$.

   (a) How are T and B_1 related?

   (b) What is the approximate width of the slice where the spin-up protons will be significantly excited (i.e., deflected significantly from the $+z$ direction)? This is the limit to your resolution: the "slice width".

   (c) How would you vary B_g to decrease the slice width? How would you vary T to decrease the slice width?

   (d) For realistic parameters in medical imaging, namely $B_0 = 1.5\,\mathrm{T}$, $T = 5\,\mathrm{msec}$, and a gradient for B_0 in the z-direction of $B_g = 0.02\,\mathrm{T/m}$, what is the approximate width of the slice?

2. *Clebsch-Gordon coefficients.*

   (a) Shankar Exercise 15.2.2, part 1: find the CG coefficients for the decomposition $\frac{1}{2} \otimes 1 = \frac{3}{2} \oplus \frac{1}{2}$. Please work this out in detail, for practice, rather than simply looking up the answers.

   (b) Shankar Exercise 15.2.3: argue that $\frac{1}{2} \otimes \frac{1}{2} \otimes \frac{1}{2} = \frac{3}{2} \oplus \frac{1}{2} \oplus \frac{1}{2}$.

3. *Combining angular momentum, again.* Suppose that electrons were spin-50 particles and had $\mu = \gamma\vec{S}$, with $\gamma \neq 0$, but still were much lighter than the nucleus. Let $\vec{J} = \vec{L} + \vec{S}$ be the total angular momentum of the electron (orbital about the nucleus + spin) and suppose $\langle L^2 \rangle = 2550\hbar^2$. Let a beam of such atoms pass through a Stern-Gerlach apparatus (standard spreading kind, not Feynman recombining kind).

   (a) If $J = 99\hbar$ and $j_z = 97\hbar$ for all the atoms entering the S-G apparatus, how many separate beams exit the S-G apparatus? What fraction of the beam will end up in the $s_z = 49\hbar$ band?

   (b) If $J = 98\hbar$ and $j_z = 98\hbar$ how many beams exit the S-G apparatus? What fraction of the beam will end up in the $s_z = 49\hbar$ band?