# PHY662 - Quantum Mechanics II
# HWK #1, Due Tues., Jan. 20, at the *start* of class

- Read Sections 6-4 and 6-5 from Feynman, et al., (also covered in class) to help with the problems and the rest of Chapter 6, just to complete this reading.

- Reading pages 373-385 of Shankar (Secs. 14-1 through 14-3) again will also be useful.

- Look on the Web for information about one-time pads, as used in cryptography. How secure are they? What fault or faults does encryption with one-time pads have? (Spend a few minutes researching, reading, and thinking - we will discuss this on Tuesday; you do not need to hand in a written answer.)

1. *The dimensionality of the group $SO(N)$, the group of proper rotations in $N$-dimensional space.* This problem is an exercise in studying the *local* structure of $SO(N)$. [The *global* topology is a different type of question: the surface of a solid torus and surface of a solid sphere each have dimension 2, but these surfaces have different topologies.] In class on Jan. 13, it was noted that the dimensionality of the rotation group in two dimensions is 1, since any rotation can be represented by a single angle $\theta \in [0, 2\pi]$, while the dimensionality of rotations in three dimensions is 3. The latter is a nice coincidence: rotations in 3-space can be denoted by rotation vectors $\vec{\theta}$. Rotation operations $R \in SO(N)$ can be defined as operations that map vectors $\vec{r} \in \mathbb{R}^N$ to vectors $\vec{r'} = R(r) \in \mathbb{R}^N$ that are *isometries,* i.e., Euclidean length is preserved, $|\vec{r'}| = |\vec{r}|$, and that preserve orientation (the inversion operator $\vec{r} \to -\vec{r}$ is not part of $SO(N)$). These length-preserving operations can be shown to be linear, that is $R(\vec{r_1}+\vec{r_2}) = R(\vec{r_1})+R(\vec{r_2})$ and $R(\lambda\vec{r}) = \lambda R(\vec{r})$, for any vectors $\vec{r}_1$ and $\vec{r}_2$ and real coefficients $\lambda$. $SO(N)$ is therefore a *subset* of the linear operators on $\mathbb{R}^N$: each $R \in SO(N)$ operating on the space of N-dimensional vectors corresponds to some $N \times N$ matrix.

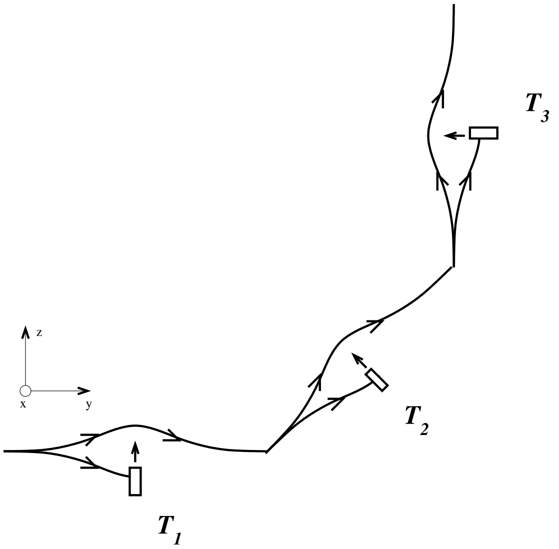
    (a) Given a choice of coordinate frame, with basis vectors $\hat{x}_i$, $i = 1 \ldots N$, let $A_{ij}$ be a matrix representing an element in $SO(N)$ in that basis. That is $A_{ij}r_j = r'_i$ is a length preserving operation, where $r_i$ ($r'_i$) are the components of $\vec{r}$ ($\vec{r'}$). Show that $\sum_k A_{ik}A_{jk} = \delta_{ij}$, where $\delta_{ij}$ is the Kronecker delta symbol.

    (b) Apply part (a) to answer the question: How many independent real variables (at most) are needed to describe the elements of $SO(N)$? (What is the dimensionality of the set of $N \times N$ matrices, i.e., how many independent real numbers are there in a general $N \times N$ matrix and how many constraints does (a) place on these elements?)

(c) Check your answer by computing the dimensionality of $SO(2)$, $SO(3)$, and $SO(4)$.

2. *Transforming amplitudes and computing probabilities in rotated frames.* In this problem, the type of S-G apparatus is that defined in Feynman, et al.

   (a) Consider the sequence diagrammed on the top of the next page, with 3 S-G apparatuses, $T_1$, $T_2$, and $T_3$, tilted at angles $\frac{\pi}{4}$ relative to each other about the $x$-axis, with the lower path (defined for each S-G apparatus separately) blocked. The arrows indicated the direction of the gradient of the magnetic field. What is the probability that an atom that passes through the first apparatus will make it through the third apparatus?



   (b) Suppose you have a sequence of $n$ S-G apparatuses, $T_i$, $i = 1 \ldots n$, each with the "lower" path blocked, with each apparatus rotated by an angle $\frac{\pi}{2(n-1)}$ about the $x$-axis, relative to the previous apparatus. (That is, a sequence as in (a), but with $n$ apparatuses, rather than 3, with the angle between $T_i$ and $T_{i+1}$ being $\frac{\pi}{2(n-1)}$.) What is the probability that an atom that is in the "up" state for the first apparatus will successfully pass through the next $n-1$ S-G apparatuses and exit from apparatus $n$? Expand your answer to order $1/n$.

   (c) How would your answer to (b) change, if at all, if you removed all of the blocks on apparatuses 2 through $n$, while keeping the beam blockers on apparatus 1 and $n$?

3. *Practice with eigenvectors, Pauli matrices, etc.* To follow the text, I suggest you carry out the exercises in Shankar's book as much as seems reasonable. The following includes some of these exercises and will give you further practice with the algebra of the rotation operators and Pauli matrices.

   (a) What are the eigenvalues of $\hat{n}\cdot\vec{S}$, as defined in Shankar Eq. (14.3.27)?

   (b) Please check that Eqs. (14.3.28a) and (14.3.28b) are eigenvectors of $\hat{n}\cdot\vec{S}$.

   (c) Verify that when $\hat{n} = \hat{x}$, Eqs. (14.3.28a) and (14.3.28b) give eigenvectors of $S_x$. Repeat this verification for the cases $\hat{n} = \hat{y}$ with $S_y$.

   (d) *Check* the answer (given in the back of the book) to Shankar's exercise 14.3.5.

   (e) The result Eq. (14.3.44) gives the explicit form for the rotation operator for a *general* rotation by an angle $\theta$about the direction $\hat{\theta}$. Mimic the derivation of this result in the specific case of $\vec{\theta} = \frac{\pi}{4}\hat{y}$ by expanding the series for the exponential more explicitly for this rotation, showing the derivation in a bit more detail. Then write out this rotation operator in the $s_z$-basis as an explicit $2 \times 2$ matrix.