Homework 9

I'm going to try and remember to cite where I pull equations from in the lectures from now on when possible. I will refer to lecture equations by their lecture and slide number, but I won't bother labeling order on a particular slide because there isn't much risk for confusion. For example if I am citing an equation from the 6th slide of lecture 9 I will label it: L.9.6, with L so it is not confused with equations within the solutions.

Problem 1

(a) Starting from the definitions

$$dr_{shell} \equiv (1 - r_S/r)^{-1/2}\, dr \tag{1}$$

$$dt_{shell} \equiv (1 - r_S/r)^{1/2}\, dt \tag{2}$$

we have

$$\frac{dr_{shell}}{dt_{shell}} = \frac{1}{(1 - r_S/r)} \frac{dr}{dt} \tag{3}$$

and combining with L 16.6:

$$\frac{dr}{dt} = -c(1 - r_S/r) \left[\frac{r_S/r - r_S/r_o}{1 - r_S/r_o} \right]^{1/2} \tag{4}$$

we have in total

$$\frac{dr_{shell}}{dt_{shell}} = \frac{1}{(1 - r_S/r)} (-c)(1 - r_S/r) \left[\frac{r_S/r - r_S/r_o}{1 - r_S/r_o} \right]^{1/2} \tag{5}$$

$$= -c \left[\frac{r_S/r - r_S/r_o}{1 - r_S/r_o} \right]^{1/2} \tag{6}$$

which for $r_o = \infty$ reduces to

$$\frac{dr_{shell}}{dt_{shell}} = -c \left(\frac{r_S}{r} \right)^{1/2} \tag{7}$$

and thus

$$\frac{dr_{shell}}{dt_{shell}} = -c \left(\frac{10}{35} \right)^{1/2} = -.53c = -1.6 \cdot 10^8 m/s. \tag{8}$$

(b) If we just take L16.6 and set $r_o = \infty$ we get

$$\frac{dr}{dt} = -c(1-r_S/r)\left(\frac{r_S}{r}\right)^{1/2} = -c(1-10/35)\left(\frac{10}{35}\right)^{1/2} = -.38c = -1.1{\cdot}10^8 m/s. \tag{9}$$

(c) Taking the above formulas and plugging in the different numbers we get

$$\frac{dr_{shell}}{dt_{shell}} = -c\left(\frac{r_S}{r}\right)^{1/2} = -c\left(\frac{10}{15}\right)^{1/2} = -.82c = -2.4\cdot 10^8 m/s \tag{10}$$

and

$$\frac{dr}{dt} = -c(1-r_S/r)\left(\frac{r_S}{r}\right)^{1/2} = -c(1-10/15)\left(\frac{10}{15}\right)^{1/2} = -.27c = -8.2{\cdot}10^7 m/s. \tag{11}$$

(d) Similarly,

$$\frac{dr_{shell}}{dt_{shell}} = -c\left(\frac{r_S}{r}\right)^{1/2} = -c\left(\frac{r_S}{r_S}\right)^{1/2} = -c. \tag{12}$$

and

$$\frac{dr}{dt} = -c(1-r_S/r)\left(\frac{r_S}{r}\right)^{1/2} = -c(1-r_S/r_S)\left(\frac{r_S}{r_S}\right)^{1/2} = 0. \tag{13}$$

Problem 2

(a) From L.16.7 we have

$$\frac{d^2 r_{shell}}{dt^2_{shell}} = -\frac{GM}{{r_o}^2}\left(1-\frac{2GM}{c^2 r_o}\right)^{-1/2} \tag{14}$$

$$= -\frac{r_S c^2}{2{r_o}^2}\left(1-\frac{r_S}{r_o}\right)^{-1/2} = -\frac{10^6 c^2}{2\,(4\cdot 10^6)^2}\left(1-\frac{10^6}{4\cdot 10^6}\right)^{-1/2} \tag{15}$$

$$= -c^2(3.6\cdot 10^{-8} m^{-1}) = -3.2\cdot 10^9 m/s^2. \tag{16}$$

Being that the acceleration is on the order of 10^8 times the acceleration of earth's gravity its probably safe to say that he's not human. At most its a flesh covered cyborg sent back in time to drop the wrench. If this is the case its probably a more advanced model than the T-800 model 101 because those have been shown to succumb to hydraulic presses which probably exert along

the lines of the force associated with the above gravitational acceleration for the mass of a T-800.
(b) The equation for tidal acceleration is

$$a_{tidal} = \frac{2GM}{r^2}\Delta r = \frac{c^2 r_S}{r^3}\Delta r \tag{17}$$

$$\approx \frac{c^2 1.0 \cdot 10^9}{(4 \cdot 10^9)^3}(1) = 1.6 \cdot 10^{-20}(1/m)c^2 = 1.4 \cdot 10^{-3} m/s^2. \tag{18}$$

Problem 3
(a) For a 5 solar mass object we have

$$r_S = \frac{2GM}{c^2} = \frac{2(6.67 \cdot 10^{-11})(5 \cdot 1.99 \cdot 10^{30})}{(3.0 \cdot 10^8)^2} = 15km. \tag{19}$$

From L17.6 for Schwarzschild coordinates we have

$$\frac{dr}{dt} = -c(1 - r_S/r)\left[\frac{r_S/r - r_S/r_o}{1 - r_S/r_o}\right]^{1/2} \tag{20}$$

$$= -c(1 - 15/16)\left[\frac{15/16 - 15/20}{1 - 15/20}\right]^{1/2} = -.05c = -1.6 \cdot 10^7 m/s. \tag{21}$$

(b) In order to calculate in shell coordinates we start with equation (3)

$$\frac{dr_{shell}}{dt_{shell}} = \frac{1}{(1 - r_S/r)}\frac{dr}{dt} \tag{22}$$

and insert (20) to obtain

$$\frac{dr_{shell}}{dt_{shell}} = \frac{1}{(1 - r_S/r)}(-c)(1 - r_S/r)\left[\frac{r_S/r - r_S/r_o}{1 - r_S/r_o}\right]^{1/2} \tag{23}$$

$$= -c\left[\frac{r_S/r - r_S/r_o}{1 - r_S/r_o}\right]^{1/2} \tag{24}$$

$$= -c\left[\frac{15/16 - 15/20}{1 - 15/20}\right]^{1/2} = -.87c = -2.6 \cdot 10^8 m/s. \tag{25}$$

(c) For the shell velocity at the event horizon we have

$$\frac{dr_{shell}}{dt_{shell}} = -c\left[\frac{r_S/r - r_S/r_o}{1 - r_S/r_o}\right]^{1/2} = -c\left[\frac{r_S/r_S - r_S/r_o}{1 - r_S/r_o}\right]^{1/2} \tag{26}$$

$$= -c\left[\frac{1-r_S/r_o}{1-r_S/r_o}\right]^{1/2} = -c \tag{27}$$

which is clearly not a function of r_o.
(d) Starting with expression L.17.5:

$$\frac{dr}{dt} = -c(1-r_s/r)\left[1-\frac{1}{\gamma^2}(1-r_S/r)\right]^{1/2}. \tag{28}$$

First we need to find the value of γ. We can do this by first setting $r = \infty$

$$\frac{dr}{dt} = -c\left[1-\frac{1}{\gamma^2}\right]^{1/2}. \tag{29}$$

and solving the above expression for γ

$$\gamma = \frac{1}{\sqrt{1-\left(\frac{1}{c}\frac{dr}{dt}\right)^2}} \tag{30}$$

which for $\frac{dr}{dt} = 99.99c$ gives $\gamma = 71$. Now inserting this back into (28) we have

$$\frac{dr}{dt} = -c(1-15/16)\left[1-\frac{1}{71^2}(1-15/16)\right]^{1/2}. = -.06c = -1.9\cdot 10^7 m/s. \tag{31}$$