Homework 6 *Note: All relevant figures are seperate*

Problem 1

(a) In general in order to integrate we treat the differences as infinitesimal changes i.e. $\Delta x \rightarrow dx$ so for our line element we have

$$dl^2 = g_{11} d\phi^2 + g_{22} d\theta^2 = sin^2(\theta) d\phi^2 + d\theta^2 \tag{1}$$

so

$$Length = l = \int_{initial}^{final} dl = \int \sqrt{sin^2(\theta) d\phi^2 + d\theta^2}. \tag{2}$$

For $\theta = \frac{\pi}{6}$ and $\phi = 0 \rightarrow \phi = \pi$ we have $d\theta = 0$ and so

$$l = \int_{initial}^{final} dl = \int_0^{\pi} \sqrt{sin^2(\theta) d\phi^2} = \int_0^{\pi} sin(\theta) d\phi = \int_0^{\pi} sin(\frac{\pi}{6}) d\phi \tag{3}$$

$$= sin(\frac{\pi}{6}) \int_0^{\pi} d\phi = sin(\frac{\pi}{6}) \phi|_0^{\pi} = \pi sin(\frac{\pi}{6}) = \frac{\pi}{2}. \tag{4}$$

(b) The shape of the path is a half-circle. From the figure we can see that the radius of the circle is simply $r = (1) sin(\frac{\pi}{6}))$ so that the length of the line is simply an arclength or $arc = (radius)$x$(angle\ in\ radians) = \pi sin(\frac{\pi}{6})$ which agrees with our prior result.

(c) This situation is different from (a) in a nontrivial way. I think about it as follows: Since we are computing the length of line there is only one degree of freedom, that is, we can only move forward or backward on the line thus in principal there is a constraint of the form

$$\phi = f(\theta) \tag{5}$$

so ϕ is not independent and I can't change θ without changing ϕ. Using the chain rule we can write

$$d\phi = \frac{f(\theta)}{d\theta} d\theta \equiv f'(\theta) d\theta \tag{6}$$

and rewrite (4) as

$$l = \int_{initial}^{final} dl = \int \sqrt{sin^2(\theta) f'(\theta) + 1} d\theta. \tag{7}$$

Now we have $\phi = \theta$ so that $f'(\theta) = 1$ or

$$l = \int_{initial}^{final} dl = \int_0^{\pi} \sqrt{sin^2(\theta) + 1} d\theta. \tag{8}$$

Good luck figuring out this integral by hand. Mathematica did it in .01 seconds and I get 3.8202.

Problem 2
By 'area element' we mean an infinitesimal piece of the area i.e.

$$dA = \sqrt{g_{11} g_{22}} d\theta d\phi \tag{9}$$

so that

$$A = \int dA = \int \sqrt{g_{11} g_{22}} d\theta d\phi = \int_0^{2\pi} d\phi \int_0^{\pi} d\theta sin(\theta)$$
$$= \phi|_0^{2\pi} (-cos(\theta))|_0^{\pi} = 2\pi\left(-(0-1)\right) = 2\pi. \tag{10}$$

Problem 3
(a) From the figure we can see that the vector rotates 180 degrees or π radians.
(b) We have *curvature* x *area* = *angle* and the area enclosed is $\frac{1}{4}$ of a whole unit sphere thus we have

$$curvature = \frac{angle}{area} = \frac{\pi}{\frac{4\pi(1)^2}{4}} = 1. \tag{11}$$

(c) I choose a path that starts at $\phi = 0, \theta = \frac{\pi}{2}$ moves up a line of longitude to the north pole at $\phi = 0, \theta = 0$ goes down another line of longitude to $\phi = \frac{\pi}{2}, \theta = \frac{\pi}{2}$ and back along a line of latitude to the starting point at $\phi = 0, \theta = \frac{\pi}{2}$. This path encloses an area $\frac{1}{8}$ of the unit sphere results in the vector rotating $\frac{\pi}{2}$ radians thus we have

$$curvature = \frac{angle}{area} = \frac{\frac{\pi}{2}}{\frac{4\pi(1)^2}{8}} = 1 \tag{12}$$

which agrees with our result from part (b).