Homework 10

I'm going to try and remember to cite where I pull equations from in the lectures from now on when possible. I will refer to lecture equations by their lecture and slide number, but I won't bother labeling order on a particular slide because there isn't much risk for confusion. For example if I am citing an equation from the 6th slide of lecture 9 I will label it: L.9.6, with L so it is not confused with equations within the solutions.

Problem 1

(a) Because it is in a circular orbit r is constant and we have

$$\frac{L}{m} = rv = r\frac{d(r\phi)}{d\tau} = r^2\frac{d\phi}{d\tau} = r^2\frac{d\phi}{dt_{shell}}\frac{dt_{shell}}{d\tau} = rv_{shell}\gamma_{shell}. \tag{1}$$

(b) By definition we have

$$v_{shell} = \frac{d(r\phi)}{dt_{shell}} = r\frac{d(\phi)}{dt_{shell}} \tag{2}$$

so that

$$dt_{shell} = \frac{rd\phi}{v_{shell}} \tag{3}$$

and integrating

$$\int_{t_i}^{t_f} dt_{shell} = \int_{\phi_i}^{\phi_f} \frac{rd\phi}{v_{shell}} = \frac{r}{v_{shell}}\int_0^{2\pi} d\phi \tag{4}$$

so that we have

$$\Delta t_{shell} = \frac{2\pi r}{v_{shell}} \tag{5}$$

(c) Again we have

$$L = rv = mr\frac{d(r\phi)}{d\tau} \tag{6}$$

rearranging we have

$$d\tau = \frac{mr^2}{L}d\phi \tag{7}$$

integrating we obtain

$$\Delta\tau = \frac{2\pi mr^2}{L}. \tag{8}$$

(d) By the definition of the shell coordinates from L.16.8

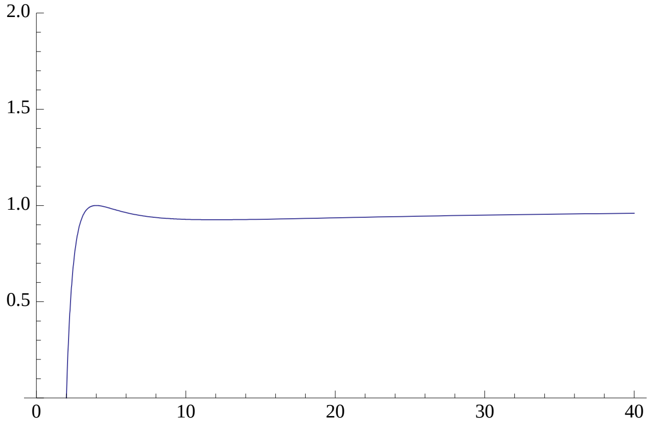
$$dt = \left(1 - \frac{r_S}{r}\right)^{-1/2} dt_{shell}. \tag{9}$$

Integrating we obtain

$$\Delta t = \left(1 - \frac{r_S}{r}\right)^{-1/2} \Delta t_{shell}. \tag{10}$$

Problem 2

(a) Let's just look at the potential for $l = 4$ first of all to get an idea of what is going on:



There is unstable orbit around $r' = 3$ and a (subtle) stable orbit around $r' = 10$. (If you keep playing around with different l's you will see that this is a general feature of this potential albeit the stable orbit is not always so clear from the plots). Thus if we want the smallest stable orbit we want the smallest r corresponding to the smallest possible angular momentum or equivalently the smallest l. From L.19.6 we have

$$r'^2 - l^2 r' + 3l^2 = 0 \tag{11}$$

the solution to which is

$$r' = \frac{l^2 \pm \sqrt{l^4 - 12l^2}}{2} = \frac{l^2 \pm l\sqrt{l^2 - 12}}{2}. \tag{12}$$

Thus the smallest possible l is $l = \sqrt{12}$. Plugging in this value for l for r we get

$$r' = 6. \tag{13}$$

Rewriting in 'unnatural' units defined on L.19.3 we have

$$r = \frac{r_S r'}{2} = 3r_S. \tag{14}$$

Problem 3

From L.19.7 we have

$$\frac{L}{m} = r v_{shell} \gamma_{shell} = \frac{8GM}{c^2}\frac{3c}{4}\frac{1}{\sqrt{1 - \left(\frac{3c/4}{c}\right)^2}} = \frac{24GM}{\sqrt{7}c} = \frac{12 c r_S}{\sqrt{7}} = 2 \cdot 10^{-18} M. \tag{15}$$

And again from L.19.7 we have

$$\epsilon \equiv \frac{E}{mc^2} = A^{1/2}\gamma_{shell} = \sqrt{1 - \frac{r_S}{r}}\frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} = \sqrt{1 - \frac{1}{4}}\frac{1}{\sqrt{1 - \left(\frac{3}{4}\right)^2}} = 1.3. \tag{16}$$

Now we have to look at a few things in order to answer whether or not the particle escapes or not. First of all the dimensionless or reduced coordinates we have

$$l = \frac{2L}{mcr_S} = \frac{2}{cr_S}\frac{12cr_S}{\sqrt{7}} = \frac{24}{\sqrt{7}} = 9.1. \tag{17}$$

Now lets look at the initial position of the satellite. The initial radial distance of the satellite in reduced coordinates is

$$r'_o = \frac{2r_o}{r_S} = \frac{2}{r_S}\frac{8GM}{c^2} = \frac{2}{r_S}4r_S = 8. \tag{18}$$

Now the local extrema are given by the roots to the stable circular orbit equation:

$$r' = \frac{9.1^2 \pm 9.1\sqrt{9.1^2 - 12}}{2} \rightarrow r'_+ = 80, r'_- = 3.1. \tag{19}$$

From the above 3 numbers $3.1 < 8 < 80$ we can see that the satellite starts out beyond the inner extrema. Now the satellite has $\epsilon = 1.3$ in reduced units. The potential at the extrema is

$$V(r'_+) = \left(1 - \frac{2}{80}\right)\left(1 + \frac{9.1^2}{80^2}\right) = .99. \tag{20}$$

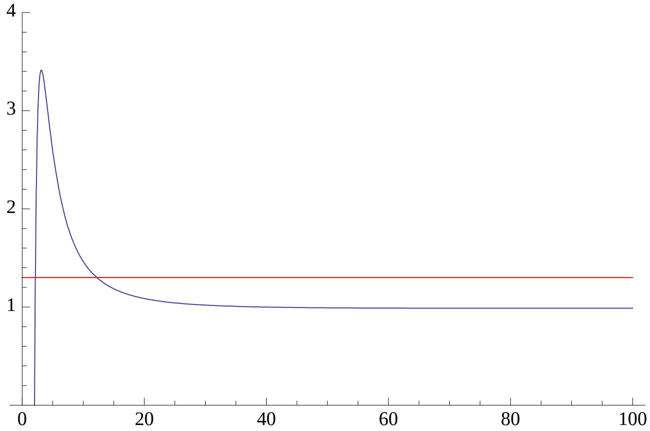
and

$$V(r'_-) = \left(1 - \frac{2}{3.1}\right)\left(1 + \frac{9.1^2}{3.1^2}\right) = 3.4 \tag{21}$$

Furthermore

$$V(r' \rightarrow \infty) \rightarrow 1. \tag{22}$$

So there is a bump at $r'_+ = 3.1$ and $r' = \infty$. Thus the particle starts out between these extrema, has bounded motion from below, and is unbounded from above. This is perhaps best illustrated with the plot below.



In the above plot the potential is blue (curved if this comes out black and white) and the dimensional less energy ϵ is in blue (straight if this comes out black and white). If you are confused about whats going on here just remember that all this funky stuff with GR and spherical coordinates doesn't matter whatso-ever in this question- by looking at the effective radial potential we have reduced the problem to a one-dimensional problem that can simply be looked at as whether or not a ball has enough energy to roll out of a valley.

Problem 4 The effective potential for a photon is

$$V_{\gamma,eff} = \frac{1}{r'^2}\left(1-\frac{2}{r'}\right). \tag{23}$$

To find the allowed circular orbits we set the first derivative equal to zero

$$\frac{\partial V_{\gamma,eff}}{\partial r'} = \frac{-2}{r'^3}\left(1-\frac{2}{r'}\right) + \frac{1}{r'^2}\frac{2}{r'^2} = \frac{-2}{r'^3} + \frac{6}{r'^4} = 0 \tag{24}$$

which yields

$$r' = 3 \tag{25}$$

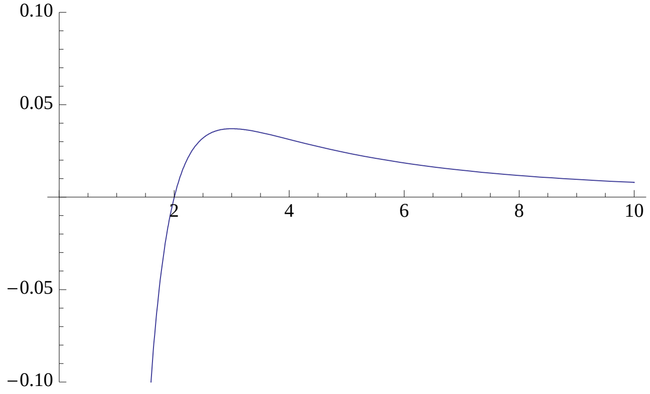
thus

$$r = \frac{2r'}{r_S} = \frac{3GM}{c^2} = 3r_\gamma \tag{26}$$

where $r_\gamma \equiv \frac{GM}{c^2}$ is the photon sphere radius. In order to find out if this orbit is stable we evaluate the second derivative of the effective potential at this radius:

$$\frac{\partial^2 V_{\gamma,eff}}{\partial r'^2}|_3 = \left(\frac{-2}{r'^3} + \frac{6}{r^4}\right)|_3 = \frac{-2}{81}. \tag{27}$$

Since $\frac{-2}{81} < 0$ the potential is concave down at this point and hence unstable. Alternatively we can look at the plot:



which makes it clear that there are no stable circular orbits and only one unstable circular orbit. Now $b = \frac{6GM}{c^2}$ so $b' = \frac{2b}{r_S} = 6$. At the peak we have

$$V_{\gamma,eff}|_{r'=3} = \frac{1}{3^2}\left(1-\frac{2}{3}\right) = \frac{1}{27}. \tag{28}$$

Now since

$$\frac{1}{b'^2} = \frac{1}{36} < \frac{1}{27} \tag{29}$$

the photon will not be captured.