PHY312 - lecture 5

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Review

- Derived Lorentz transformations relating coordinates of events as seen by different inertial observers.
- Saw how to add velocities in special relativity.
- Examples
- Today: talk about vectors in spacetime.
- Energy-momentum vector ...

Vectors in space

- Consider motion of some particle in 2d space.
- Use two different coordinate systems (x, y) and (x', y') which are rotated by angle θ.
- Position vector

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j}$$

where $\mathbf{i}$ and $\mathbf{j}$ are unit vectors in x and y directions.

- Equally same position can be written

$$\mathbf{r} = x'\mathbf{i}' + y'\mathbf{j}'$$

Transformation between coordinates

- Taking dot products of first eqn. with respect to $\mathbf{i}'$ leads to
$$x' = x(\mathbf{i}.\mathbf{i}') + \mathbf{y}(\mathbf{j}.\mathbf{i}')$$
But $\mathbf{i}.\mathbf{i}' = \cos(\theta)$ and $\mathbf{j}.\mathbf{i}' = \sin(\theta)$ so that the transformation of the coordinates may be written
$$\begin{aligned} x' &= \cos(\theta)x + \sin(\theta)y \\ y' &= -\sin(\theta)x + \cos(\theta)y \end{aligned}$$
- Notice that the length of the position vector $x^2 + y^2 = (x')^2 + (y')^2$ is **invariant** with respect to transformation between different (rotated) coordinate systems (although the component values x and y are certainly not).

Definition of vectors

- Can define 2d (space) vector as set of 2 numbers (x, y) which transform according to this rule as coordinate system is rotated.
- This rule is determined by requiring that the length $x^2 + y^2$ same in all frames ...
- Sound familiar ?
- Natural way to map the idea of an invariant spacetime interval under LT into the language of 2d rotations ...

The correspondance

- (Minus) spacetime distance squared $x^2 - c^2t^2$=$x^2 + T^2$ if $T = ict$.
- Consider previous construction with $y \rightarrow T$.
- Set $x' = 0$. Rotation angle $\tan\theta = x/T$
- Thus $\tan\theta = -i\frac{v}{c}$. Using $\frac{1}{1+\tan^2(\theta)} = \cos^2(\theta)$ find

$$\cos(\theta) = \frac{1}{\sqrt{1 - \left(\frac{v}{c}\right)^2}} = \gamma$$

$$\sin(\theta) = \tan(\theta)\cos(\theta) = \gamma i \frac{v}{c}$$

- Coordinate transformation –> LT!!

Spacetime vectors

- Define spacetime vector (sometimes called 4-vector) as a vector that transforms same way as the spacetime coordinate vector.
- Has 3 space and 1 time component. (a_t, a_x, a_y, a_z).
- Length $a_t^2 - a_x^2 - a_y^2 - a_z^2$ same in all inertial FOR.
- Components transform using LT.
- If theory formulated in terms of such vectors we are guaranteed that equations will look same in all inertial FOR !
- Can construct things everyone agrees on ..

Energy-moemntum vector

- Consider worldline of particle in spacetime.
- Any small portion of it looks like straight line with components (in some FOR) $(c\Delta t, \Delta x, \Delta y, \Delta z)$
- This is simplest example of spacetime vector.
- Points along worldline at that point.
- Calculate proper time for this small displacement $\Delta\tau$.
- Consider vector

$$P = m_0 \left(\frac{c\Delta t}{\Delta\tau}, \frac{\Delta x}{\Delta\tau}, \frac{\Delta y}{\Delta\tau}, \frac{\Delta z}{\Delta\tau} \right)$$

Why ?

- What is m_0 ? What is the physical interpretation of this vector ?
- $d\tau^2 = dt^2(1 - \frac{v^2}{c^2})$. Consider $v/c \to 0$. Spatial components look like velocity times m_0. Thus treat m_0 as rest mass (mass measured in frame in which is it at rest)
- Thus spatial components are relativistic generalization of momentum!
- But what about the time component $m_0 c \frac{dt}{d\tau}$?

$E = mc^2$

- Consider small v. Taylor expand the square root

$$m_0 c \frac{dt}{d\tau} = m_0 c \left(1 + \frac{1}{2}\frac{v^2}{c^2} + \ldots\right)$$

  Contains the Newtonian kinetic energy K/c plus constant.

- It is a relativistic generalization of energy of motion.
- But notice it has a value even when at rest $m_0 c^2$! Rest energy. But since c is a constant shows that this rest energy is just a measure of mass.
- Famous equivalence of mass and energy $E = m_0 c^2$. Large amount of energy from small mass ...

Summary

- Rederived LT by analogy with 2d spatial rotations. Concept of spacetime vector. Invariance of interval from invariance of length.
- Laws of physics should be written in terms of such vectors. Ensures principle of relativity ...
- Simplest example of such a vector – energy-momentum vector. Gives relativistic generalization of energy and momentum.
- Shows equivalence of mass and energy.