Relativity and Cosmology

lecture 21

Recap lecture 20

- Photon trajectories in Schwarzschild.
- Newtonian limits for energy/acceleration in Schwarzschild.

Completed discussion of motion static, spherically symmetric spacetime
Constants of motion, different FOR, event horizon, orbits etc
Spinning black holes ? Not this time ...

Cosmology

- Large scale dynamics of Universe thought to be governed only by gravitational forces. In the framework of GR we thus expect the large scale features of the Universe to be described by a spacetime metric which is a solution of the field equations of GR.
- Solutions of Einstein's equations represent all possible spacetimes. Which ones describe our Universe ? This is the subject of cosmology.
- To proceed need to make simplifying assumptions:
  - What type of T appears on RHS of field equations ?
  - What symmetries should we assume for metric describing Universe at large scales ?

Symmetries of spacetime

- Sun is one of about 100 billion stars which are clustered together into Milky Way (about 100 thousand light years across). This is a typical scale for a galaxy. Between galaxies there are voids in space. Galaxies often group into clusters and superclusters - sometimes these appear to be correlated in position.
- However, if you average over large enough distances the Universe appears remarkably homogeneous and isotropic (looks same at all distances and directions).
- Leads to a *cosmological principle* which basically says that the Universe looks the same from all points within it.

Consequences

- This assumption implies that spacetime is made up of 3d spaces each associated with a different instant of time. Furthermore this space must have constant spatial curvature (since no point is distinguished). There are then 3 possibilities for this space.
  1. Flat Euclidean space $R = 0$. Parallel geodesics never cross. Infinite in size.
  2. The 3d sphere S^3 with positive (spatial) curvature $R > 0$. Initially parallel geodesics converge. Finite in size.
  3. 3d hyperboloid. Points in 4d flat space such that $t^2 - x^2 - y^2 - z^2 = 1$. Diverging geodesics. Infinite in size. $R < 0$.

LHS – FRW metric

- All these possibilities can be realized from the metric (Friedmann, Robertson, Walker metric)

$$ds^2 = dt^2 - a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)\right]$$

where $k=0$, $k=1$ and $k=-1$ label the three spatial topologies. This form of the metric also incorporates the isotropy of each spatial slice at fixed t.

- Which topology ? Until recently this was unclear - now new observations appear to select the first possibility – a spatially flat Universe on the largest scales ($k=0$). (COBE/WMAP data)
- Note: single scale factor $a(t)$ governs behavior of Universe - typical distance between galaxies ...

RHS - matter

- RHS of Einstein's equations is energy-momentum $T_{\mu\nu}$. Must be given in terms of the (spatially constant) energy density and pressure of matter and radiation.
- The simplest form for these consistent with our cosmological principle is

$$T_{\mu\nu} = (\rho + p)\, u_\mu u_\nu - p g_{\mu\nu}$$

where ρ and p are the density and pressure of a perfect fluid (we have set $c = 1$)

Cosmological equations

- Finally Einstein's equations become

(1) $$3\frac{\left(\left(\frac{da}{dt}\right)^2+k\right)}{a^2} = 8\pi G\rho+\Lambda$$

(2) $$\frac{2a\frac{d^2a}{dt^2}+\left(\frac{da}{dt}\right)^2+k}{a^2} = -8\pi Gp+\Lambda$$

- The term involving Λ is called the **cosmological constant term** and was introduced by Einstein so that the equations of GR admitted static solutions.

More on Λ

- However in 1929 Hubble announced that all the galaxies were receeding from each other with a velocity proportional to their separation - just what you would have expected if $a(t)$ were increasing with time ! Thus Einstein regarded the introduction of this new term as his "Greatest Mistake".
- However, recent modern observations on the apparent acceleration of the distant supernovae appear to favor such a term afterall. The nature and origin of this *dark energy* is a source of much work right now. Also, natural in QM - but why do small ?

Continued

- Can rewrite the two cosmological equations as

$$\frac{d^2a/dt^2}{a} = -\frac{4\pi G}{3}(3p+\rho) + \frac{\Lambda}{3}$$

an equation for the cosmic acceleration and

$$\left(\frac{da/dt}{a}\right)^2 = H^2(t) = \frac{8\pi G}{3}\rho - k/a^2 + \frac{\Lambda}{3}$$

where H is the Hubble “constant” governing the velocity of the scale factor. Notice that a non-zero value of H indicates an expanding Universe as observed by Hubble.

Redshift

- To clarify let us define the **redshift** z as

$$1 + z = \frac{a(t_1)}{a(t_0)}$$

where $t_1 > t_0$. Hence in an expanding Universe $z > 0$. Suppose now that t_0 and t_1 are "close". Then we can expand $a(t_0) = a(t_1) + da/dt|_{t=t_1}\,(t_1 - t_0)$ Thus

$$z = \frac{da/dt}{a}\delta t$$

But for radially moving light rays in a flat FRW universe $\delta t = a(t)\delta r$ find $z = Ha(t_1)\Delta r$. This is Hubble's law! The red shift is proportional to the distance.

Equation of state

- To proceed further conventional to parametrize the so-called *equation of state* for the matter as $p = w\rho$.
- Radiation $\omega = 1/3$ (or extremely relativistic matter) and for (cold) non-relativistic matter matter $\omega = 0$. The cosmological constant can even be thought of as a peculiar type of matter with $w = -1$.
- Notice that we can only get a accelerating expansion (which we now observe) if $w < -1/3$ which can be done with either a pure cosmological constant or some other exotic form of matter with $w < -1/3$. It *cannot* be achieved with ordinary matter (dark energy problem)

Universes with $p = 0$

- Consider a Universe made up predominately of non-relativistic matter with $p = 0$. The pressure equation can be rewritten as

$$\frac{d}{dt}\left(a\,(da/dt)^2 + ka - \frac{\Lambda}{3}a^3\right) = 0$$

Using the first cosmological equation we then find the constant

$$C = \frac{8\pi G}{3}a^3\rho$$

It clearly has an interpretation as the (conserved) amount of mass in a Universe of radius a.

Evolution

- The equation governing the Hubble parameter now looks like

$$\left(\frac{da}{dt}\right)^2 = \frac{C}{a} + \frac{\Lambda}{3}a^2 - k$$

From now on we will consider the (experimentally most relevant case $k = 0$)

- If we set $\Lambda = 0$ we find

$$a(t) \sim t^{\frac{2}{3}}$$

This is called the Einstein-de Sitter model.

- Furthermore, if $\Lambda < 0$ then we can even find a static Universe with $da/dt = 0$ – this was the model first constructed by Einstein (and the reason Λ was introduced).

$\Lambda < 0$

- If $\Lambda < 0$ we see that $\frac{da}{dt} = 0$ for $a = (3C/\Lambda)^{\frac{1}{3}}$. Thus the expansion will halt at some point and recontraction will occur – an oscillating Universe. We can find the exact solution in this case too

$$a^3 = \frac{3C}{2 - \Lambda}\left(1 - \cos\sqrt{3\,(-\Lambda)}t\right)$$

This solution is termed *anti-de Sitter space*.

$\Lambda > 0$

- Notice that for positive Λ we can find solutions of the form $a(t) \sim \exp \sqrt{\frac{\Lambda}{3}} t$ – the spacetime is called *de Sitter* and describes one possible type of *inflationary* Universe.
- The best fit to current observation demands a small cosmological constant of this sign.

General situation

- In general we might want the solution to the cosmological equations for matter with a general equation of state w. If w is constant it is not hard to show that the 2 cosmological equations imply that

$$\frac{dH}{dt} + \frac{1+3\omega}{2} H^2 = 0$$

where $H = 1/ada/dt$ the Hubble velocity. Using this one can show:

$$a(t) = a_0 \left(\frac{t}{t_0} \right)^{\frac{2}{3(1+w)}}$$

where a_0 is current scale factor (and t_0 age of Universe)