Relativity and Cosmology

lecture 20

Recap lecture 19

- Discussed effective potential for motion of massive bodies in black hole Schwarzschild spacetime.
  - No stable orbits for small L.
  - Capture by singularity inevitable for large enough E.
  - Bound orbits possible for smaller $\epsilon = E/mc^2$ and large enough $2L/(mcr_S) \geq \sqrt{12}$. Generically correspond to precessing ellipse
  - Stable/unstable circular orbits possible by tuning E with L.

Today

- Example - escape from a black hole
- Massless particle orbits.
- More on the Newtonian limit.

Escape

- Satellite fired at 90^0 to radial direction by an observer at distance $r_o = 5r_S$ from black hole. The measured (local) speed is $c/2$.
- Angular momentum and energy are given by

$$L/m = r_o \gamma_{shell} v_{shell}$$

i.e

$$l = 2L/(mcr_S) = 5.75$$

$$E/mc^2 = A(r_o)^{1/2} \gamma_{shell} = 1.033$$

- So, has enough energy to escape to infinity if launched radially ... but it has a non-zero L so that is not the case. Need to look at effective potential ...

Escape II

- Construct effective potential. Find circular orbits.

$$r_{unstable} = 3.34 \ \ r_{stable} = 29.72$$

- Height of effective potential at $r = r_{unstable}$ is $V_{max} = 1.09$
- Thus $\epsilon = 1.033 < V_{max}$ and satellite escapes to infinity.

Photon motion

- For massive bodies used extremal ageing to find constants of motion and from there solved for the motion.
- For light $\Delta\tau = 0$ always - tricky ..
- Consider massive particle and carefully take limit $m \rightarrow 0$.
- First replace $dr/d\tau$ by dr/dt in radial equation.

$$(\frac{dr}{dt})^2 = (\frac{dr}{d\tau})^2(\frac{d\tau}{dt})^2$$

with

$$\frac{d\tau}{dt} = A(r)/(E/mc^2)$$

Continued

- Find:

$$\frac{1}{c^2}(\frac{dr}{dt})^2 = (1 - r_S/r)^2 - (1 - r_S/r)^3 \left((\frac{mc^2}{E})^2 + (\frac{Lc}{E})^2 \frac{1}{r^2} \right)$$

$$\frac{1}{c}\frac{d\phi}{dt} = \frac{Lc}{E}\frac{(1 - r_S/r)}{r^2}$$

- Take limit $m \to 0$.

$$\frac{1}{c}\frac{dr}{dt} = \pm(1 - r_S/r)\left[1 - (1 - r_S/r)\frac{(Lc/E)^2}{r^2}\right]^{\frac{1}{2}}$$

Rescaling etc

- Notice that Lc/E is constant for massless photons. Impact parameter $b = cL/E$.
- Defining $r = r' r_S/2$, $t' = 2tc/r_S$ and $b' = 2b/r_S$ find

$$\frac{dr'}{dt'} = \pm(1 - \frac{2}{r'}) \left[1 - (1 - \frac{2}{r'})\frac{b'}{r'}^2\right]$$

and

$$\frac{d\phi}{dt'} = b' \frac{1}{r'^2}(1 - \frac{2}{r'})$$

Photon effective potential

- RHS of radial equation possesses no constant like ϵ. Hard to think of an effective potential formulation.
- However, $\frac{dr_{shell}}{dt_{shell}}$ has one

$$(\frac{dr_{shell}}{dt_{shell}})^2 = 1 - (1 - 2/r)b^2/r^2$$

or

$$\frac{1}{b^2}(\frac{dr_{shell}}{dt_{shell}})^2 = \frac{1}{b^2} - V(r)$$

where

$$V(r) = (1 - 2/r)\frac{1}{r^2}$$

Orbits etc

- $V(r)$ has a maximum at $r = 3/2r_S$.
- Horizontal lines show values of $1/b^2$. Light with impact parameter b such that $1/b^2$ is bigger than peak will be captured by black hole.
- Conversely light with $1/b^2$ less than peak will scatter off black hole (bend) but leave.
- Notice possibility of unstable circular orbit for light with $1/b^2 = peak = 2.6r_S$. May go to infinity or be captured subsequently.