Relativity and Cosmology

lecture 19

Recap lecture 18

- Discussed general motion in Schwarzschild spacetime.
- Derived conservation of angular momentum. Key to solving for general motion.

Equations of motion

- Choose small $\Delta\tau$ - use equations to go from $(r, \phi) \rightarrow (r + \Delta r, \phi + \Delta\phi)$.
- Use these as new (r, ϕ) and do again. Keep iterating. General path $(r(\tau), \phi(\tau))$.
- Computer simulation ...
- Setting $r' = 2r/r_S$, $\Delta\tau' = 2\Delta\tau c/r_S$, $\epsilon = E/mc^2$ $l = 2L/(mcr_S)$ find

$$\Delta r' = \pm \left[\epsilon^2 - (1 - \frac{2}{r'})(1 + \frac{l^2}{r'^2})\right]^{\frac{1}{2}} \Delta\tau'$$

$$\Delta\phi = \frac{l}{r'^2} \Delta\tau'$$

Effective potential

- Rewrite (dimensionless) r-equation

$$(\frac{dr}{d\tau})^2 = \epsilon^2 - V(r)^2$$

where

$$V(r) = (1 - \frac{2}{r})(1 + \frac{l^2}{r^2})$$

Called effective potential.

- Different regimes; if l small monotonic inward moving particle is drawn inevitably to $r = 0$. For larger l other scenarios possible.

Turning points

- To understand what can happen consider the intersection of the curve $y = V(r)$ with $y = \epsilon$.
- Radial velocity given by difference in squares of two curves.
- Motion confined to regions where $\epsilon > V$.
- When $\epsilon = V \ v = 0$. In general place where sign of $\frac{dr}{dt}$ changes.
- Maximum or minimum in distance.
- In general motion will be such as have r oscillate between these limits elliptical orbit (with precession).
- If ϵ large – no intersections - straight capture.
- If tune ϵ so just one intersection – circular orbit.

Circular orbits

- Occur for maxima/minima of V. Find by setting $\frac{dV^2}{dr} = 0$
- Circular orbits occur for solutions of

$$3l^2 - l^2 r + r^2 = 0$$

- No circular orbits if angular momentum too small.

Initial conditions

- Determining L and E. Imagine a shell observer launching a satellite with a certain speed perpendicular to the radial direction. Can we use this info to determine L and E and hence predict the motion ?
- $E/mc^2 = A\frac{dt}{d\tau} = A\frac{dt}{dt_{shell}}\frac{dt_{shell}}{d\tau}$ This is then $E/mc^2 = AA^{-\frac{1}{2}}\gamma_{shell}$. Thus, $E/mc^2 = A^{\frac{1}{2}}\gamma_{shell}$.
- Similarly, $L/mc = r_0\frac{dt_{shell}}{d\tau}\left(r_0\frac{d\phi}{cdt_{shell}}\right)$. Thus $L/m = r_0\gamma_{shell}v_{shell}$.
- This can be trivially generalized to arbitrary initial directions of motion.