

Relativity and Cosmology

lecture 18

Recap lecture 17

- Discussed radial motion of massive and massless bodies in Schwarzschild spacetime
- Used several frames; global “far away” frame, local shell observer and freely falling observer.
- Locally laws of SR apply - light travels at speed c , massive bodies with $v < c$, acceleration, energy follows Newton for small v , $r > r_S$
- Globally things different. For BH - event horizon, singularity
- What about more general motion ? orbits ?

More constants of motion

- In analysing radial motion key was to find a conserved quantity - the energy-at-infinity
- For orbital motion we need to find another such conserved quantity – relativistic generalization of **angular momentum**.
- This when combined with metric will allow us to solve the motion

Angular momentum a la Newton

- For linear motion we know that in absence of external forces acceleration is zero and hence **momentum** $p = mv$ is constant.
- If switch on a force F this is not true. However, if force is purely radial $F(x, y, z) = F(r)$ there is still a conserved quantity – angular momentum $L = mvr = mr^2 \frac{d\theta}{dt}$
- Why ? Take cross product of $F = m \frac{dv}{dt}$ with r . $F \times r = 0$.
Hence $m \frac{d(v \times r)}{dt} = 0$.
- Thus $mv \times r = \text{constant}$. In polar coords $v = r \frac{d\theta}{dt}$.

Example

- For simple circular motion with radius a , $L = ma^2\omega$ where $\omega = \frac{d\theta}{dt}$ **must** be constant if radius a is constant.

Schwarzschild metric

- Again, principle of extremal ageing (geodesic motion) allows us to find angular momentum

$$L = mr^2 \frac{\Delta\theta}{\Delta\tau}$$

- Differs only from classical expression by using proper time (as expected) in denominator

Derivation

- Consider a motion comprising two sections A and B delineated by three events with coordinates
 - $(r + \Delta r, 0) \rightarrow (r, \phi)$ (the A section with mean r-coordinate r_A and elapsed proper time τ_A).
 - $(r, \phi) \rightarrow (r - \Delta r, \Phi)$ (the B section with mean r-coordinate r_B and elapsed proper time τ_B).
- The final angle Φ is considered fixed and only the intermediate angle ϕ will be varied.

continued ...

- Setting $\frac{d\tau}{d\phi} = 0$ and using metric we find



$$r_A^2 \frac{\phi}{\tau_A} = r_B^2 \frac{\Phi - \phi}{\tau_B}$$

- Thus we see that $r^2 \frac{d\phi}{d\tau} = \text{constant}$ and can be identified with a (conserved) angular momentum (per unit mass) for motion in a spherically symmetric spacetime.
- Notice similarity to derivation of energy ...

Remarks

- Notice that we now have two conserved quantities – energy $E = mc^2 A(r) \frac{dt}{d\tau}$ and angular momentum $L = mr^2 \frac{d\phi}{d\tau}$.
- The fundamental reason why these two quantities do not change with time is related to the fact that the components of the Schwarzschild metric do not depend **explicitly** on either time t or angle ϕ . This independence of the metric on certain coordinates is referred to as a symmetry (isometry)
- It is intimately connected to the existence of conservation laws.

General motion

- Starting from some initial position (r, ϕ) and the constants L and E we can imagine computing changes in t and ϕ using the equations

$$\Delta t = \frac{(E/mc^2)}{1 - 2GM/c^2r} \Delta\tau$$

$$\Delta\phi = \frac{L/m}{r^2} \Delta\tau$$

- Using the form of the Schwarzschild metric leads to an expression for the change in r-coordinate:

$$\Delta r = \pm \left[(E/mc^2)^2 - \left(1 - \frac{2GM}{c^2r}\right) \left(1 + \left(\frac{L/m}{cr}\right)^2\right) \right]^{\frac{1}{2}} \Delta\tau c$$

Comments

- Choose small $\Delta\tau$ - use equations to go from $(r, \phi) \rightarrow (r + \Delta r, \phi + \Delta\phi)$.
- Use these as new (r, ϕ) and do again. Keep iterating. General path $(r(\tau), \phi(\tau))$.
- Computer simulation ...
- Setting $r' = r/r_S$, $\Delta\tau' = \Delta\tau c/r_S$, $\epsilon = E/mc^2$
 $l = L/(mcr_S)$ find

$$\Delta r' = \pm \left[\epsilon^2 - \left(1 - \frac{1}{r'}\right) \left(1 + \frac{l}{r'^2}\right) \right]^{\frac{1}{2}} \Delta\tau'$$

$$\Delta\phi = \frac{l}{r'^2} \Delta\tau'$$