Realtivity and Cosmology

lecture 17

Recap lecture 16

- Energy:

$$E_{GR} = mc^2 \left(1 - \frac{2GM}{c^2 r}\right) \frac{\Delta t}{\Delta \tau}$$

- Radial speed:

$$\frac{\Delta r}{\Delta t} = -c \left(\frac{2GM}{c^2 r}\right)^{\frac{1}{2}} \left(1 - \frac{2GM}{c^2 r}\right)$$

- Shell observer sees just

$$\frac{\Delta r_{shell}}{\Delta t_{shell}} = -c \left(\frac{2GM}{c^2 r}\right)^{1/2}$$

Energy measured by shell observer

- $$E_{shell} = \frac{E}{\sqrt{1 - \frac{2GM}{c^2 r}}}$$

  where $E = mc^2$ is energy measured by far away observer.
- As $r \rightarrow R_S$ energy available to local observer becomes infinite!

Time to crunch

- Once pass event horizon object will reach $r = 0$ in finite proper time

$$\frac{\Delta r}{\Delta \tau} = -c \left(\frac{2GM}{c^2 r} \right)^{1/2}$$

- Integrate to find total proper time $\tau_C = \frac{4}{3c} \frac{GM}{c^2}$
- What is τ_C for solar mass black hole ?

More examples

- What if we throw the object with some initial speed into the black hole.
- Conservation of energy leads to

$$\gamma = (1 - r_S/r)\frac{dt}{d\tau}$$

where γ corresponds to the initial value of $E/(mc^2)$ for an object in (radial) motion (inwards).

- Substitute into this the expression for $d\tau$ from the Schwarzschild metric and we find

$$\frac{dr}{dt} = -c(1 - r_S/r)\left[1 - \frac{1}{\gamma^2}(1 - r_S/r)\right]^{\frac{1}{2}}$$

- We see that these limiting speeds remain the same

Another variation

- Dropping from a finite r-coordinate.
- Answer:

$$\frac{dr}{dt} = -c\,(1 - r_S/r)\left[\frac{r_S/r - r_S/r_o}{1 - r_S/r_o}\right]^{\frac{1}{2}}$$

and a similar expression for the shell velocity.

- Notice that the shell speed approaches the speed of light at the horizon independent of the initial r-coordinate r_o !!

corrections to Newton

- Consider some shell observer at r-coordinate r_o. Differentiate the above result with respect to t_{shell} (and reexpress the expression as a function only of shell quantities) to derive an expression for the (locally defined) gravitational acceleration in GR $\frac{d^2 r_{shell}}{dt^2_{shell}}$

$$\frac{d^2 r_{shell}}{dt^2_{shell}} = -\frac{GM}{r_o^2}\left(1 - 2GM/r_o\right)^{-\frac{1}{2}}$$

- Newton approximates GR for plunging object for small v_{shell} and GM/r.

Continued

- From $E/m = (1 - r_S/r)\frac{dt}{d\tau}$ subsitute $\frac{dt}{d\tau} = \frac{dt}{dt_{shell}}\frac{dt_{shell}}{d\tau}$. Use the expression you know for $\frac{dt}{dt_{shell}}$. Use also the expression $d\tau^2 = dt^2_{shell} - dr^2_{shell}/c^2$ to find $\frac{dt_{shell}}{d\tau}$. Hence show that for small enough v_{shell}/c and r_S/r that

$$E/m \sim 1 + \frac{1}{2}v^2_{shell} - \frac{GM}{r}$$