PHY312 - lecture 16

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Summary of lecture15

- Exterior spacetime to spherically symmetric, time independent, non-spinning gravitational source is given by Schwarzschild metric

$$\Delta s^2 = A(r)c^2\Delta t^2 - \frac{1}{A(r)}\Delta r^2 - r^2\Delta\theta^2$$

  with $A(r) = 1 - \frac{r_S}{r}$
- $r_S = 2GM/c^2$ Schwarzschild radius. Event horizon.
- Discussed motion of photon relative to far away observer, shell observer, free fall observer.

What about massive particle?

- In SR learnt that correct formula for energy is

$$E_{SR} = mc^2 \frac{\Delta t}{\Delta \tau}$$

  It is conserved.

- In GR the correct generalization is

$$E_{GR} = mc^2 \left(1 - \frac{2GM}{c^2 r}\right) \frac{\Delta t}{\Delta \tau}$$

  It is also conserved. Energy-at-infinity.

- Key to analysing radial motion.

Principle of maximal ageing

- Consider flat spacetime first. Watch free particle move from event $(0,0)$ thru (t,x) to final (T,X).
- Think of initial and final events as fixed and (t,x) as variable.
- Call two parts of motion A and B.
- Principle of maximal ageing says

$$\frac{d\tau}{dt} = 0$$

  where $\tau = \tau_A + \tau_B$
- This yields

$$\frac{1}{2\tau_A}\frac{d\tau_A^2}{dt} + \frac{1}{2\tau_B}\frac{d\tau_B^2}{dt} = 0$$

Continued

- In SR we have

$$c^2\tau_A^2 = c^2t^2 - x^2 \quad c^2\tau_B^2 = c^2(T-t)^2 - (X-x)^2$$

- leading to

$$\frac{t}{\tau_A} = \frac{T-t}{\tau_B}$$

- Thus the path followed by a free particle is one in which the quantity t/τ is *conserved* i.e fixed.
- This is another way to see that $E/m = \frac{dt}{d\tau}$ is correct definition of energy.
- Advantage: can now generalize to radial motion in Schwarzschild metric

Finally ..

- Think of events $(T, r-\Delta r)$, (t, r) and $(0, r+\Delta r)$ corresponding to the geodesic motion of a test particle in the spacetime
- Require again that

$$\frac{1}{2\tau_A}\frac{d\tau_A^2}{dt} + \frac{1}{2\tau_B}\frac{d\tau_B^2}{dt} = 0$$

- But now τ_A etc depends on metric ...
- Set $r+\Delta r = r_A$ and $r-\Delta r = r_B$. Find

$$A(r_A)\frac{t}{\tau_A} = A(r_B)\frac{T-t}{\tau_B}$$

Falling into a BH

- Consider particle released from rest at infinity. Energy is mc^2 and is conserved.
- Conservation of energy:

$$\left(1 - \frac{2GM}{c^2 r}\right)^2 \Delta t^2 = \Delta\tau^2$$

- Combine with metric:

$$\frac{\Delta r}{\Delta t} = -c\left(\frac{2GM}{c^2 r}\right)^{\frac{1}{2}}\left(1 - \frac{2GM}{c^2 r}\right)$$

Interpretation

- Notice: as for light speed goes to zero for far away observer as $r \to r_S$.
- Thus to far away observer objject never crosses the horizon ! But light is redshifted away so we don't see this ...
- What about a shell observer ? Using the expressions $\Delta r_{shell} = A^{-1/2}\Delta r$ and $\Delta t_{shell} = A^{1/2}\Delta t$ find

$$\frac{\Delta r_{shell}}{\Delta t_{shell}} = -c\left(\frac{2GM}{c^2 r}\right)^{1/2}$$

To him/her the object approaches the speed of light!

Energy measured by shell observer

- $$E_{shell} = \frac{mc^2}{\sqrt{1 - (\frac{v}{c})^2}}$$

- Using for $v = \Delta r_{shell}/\Delta t_{shell}$ find

$$E_{shell} = \frac{E}{\sqrt{1 - \frac{2GM}{c^2 r}}}$$

where $E = mc^2$ is energy measured by far away observer.

- As $r \to R_S$ energy available to local observer becomes infinite!

Conclusions

- Again, there are no contradictions here. Neither energy nor velocity are invariant physical quantities in SR or GR.

Time to crunch

- Once pass event horizon object will reach $r = 0$ in finite proper time.
- Need $\frac{dr}{d\tau} = \frac{dr}{dt}\frac{dt}{d\tau}$
- Conservation of energy means

$$\frac{dt}{d\tau} = 1/A(r)$$

- So find

$$\frac{\Delta r}{\Delta \tau} = -c\left(\frac{2GM}{c^2 r}\right)^{1/2}$$

- Integrate to find total proper time $\tau_C = \frac{4}{3c}\frac{GM}{c^2}$
- What is τ_C for solar mass black hole ?