PHY312 - lecture 14

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Summary so far ...

- POGR and POE guided Einstein to propose a radical new way of thinking about (tidal) gravity as curved spacetime.
- Curvature related to distribution of energy-momentum – specified in field equations of GR.
- Test particles follow geodesics (closest thing to straight lines on such a space)
- For small velocities/curvature – reduce to Newtonion picture.
- Early successes - gravitational red shift, perihelion of Mercury, bending of light by Sun – used approx solutions.

Exact solutions ...

- Exact solutions rare and hard to find. Numerical solutions challenging (eg binary black hole collision project, needs supercomputer level effort)
- However in some very simple cases possible - eg Schwarzschild solution. First exact solution to GR. Describes spacetime outside spherically symmetric, static mass distribution. 1915.
- Applies to Sun, Earth, neutron star, black hole, ...

Schwarzschild metric

- In (2+1) polar coordinates (r, θ) (z direction suppressed) metric is

$$\Delta s^2 = A(r) c^2 \Delta t^2 - \frac{1}{A(r)} \Delta r^2 - r^2 \Delta \theta^2$$

- Where $A(r) = 1 - \frac{2GM}{c^2 r}$
  - Unique spherically symmetric, time independent solution.
  - Yields flat (Minkowski) space for $r \to \infty$ and/or $M \to 0$.
  - r and t are “far away” coordinates.

Physical interpretation

- Consider two events in (r, t, θ) frame with $\Delta\theta = \Delta t = 0$.
- Spacelike separation $\Delta s = \Delta r_{\text{shell}} = \frac{\Delta r}{A(r)^{\frac{1}{2}}}$
- This is the spatial distance an observer at r-coordinate r would measure if he dropped a plumb line radially inward a small distance.
- Physical. Analogous to value of proper time measured by a clock in comoving FOR in SR (timelike interval).
- Such an observer called a shell observer and this distance is a shell distance.
- Notice: to stay at a fixed r he must be accelerating outward. Not in a FFF.

Examples

- What is the distance between the two shells with r coordinate r_1 and r_2 if:
  - $r_1 = 695,980km$ and $r_2 = 695981km$ for Sun with mass $2 \times 10^{30}kg$
  - $r_1 = 4km$ and $r_2 = 5km$ for solar mass black hole

  Use $G = 6.7 \times 10^{-11}\ m^3/kgs^2$
- Should see that $\frac{\Delta r_{shell}}{\Delta r}$ is quite differentin 2 cases.
- Large deviations direct measure of space(time) curvature !

Time coordinate

- Now consider 2 events at same r coordinate (and angle θ).
- What is proper time measured between these events ?

$$\Delta\tau = A(r)^{\frac{1}{2}}\Delta t$$

- Proper time < time measured by far away observer t. This is slowing of clocks in a gravitational field (now done exactly). It is another manifestation of spacetime curvature.
- Consider light propagating outward. Number of wavecrests is fixed. Frequency hence changes as time between crests changes. Gravitational redshift ..
- $\tau = t_{shell}$ time measured by stationary observer at r.

Examples again

- What is gravitational redshift $\frac{\delta f}{f} = A^{-1/2}$ for light emitted from surface of Sun and from $r = 4km$ from solar mass black hole ?

Note Metric varies continuously in space(time).
All these Δr Δt's etc should be differentials (infinitesmally small or local).

Event horizon

- Notice something odd. The time measured by a shell observer vanishes if $A = 0$ or

$$r = r_S = 2GM/c^2$$

- Schwarschild radius. Depends only on mass M.
- Infinite redshift at $r = r_S$ – all light emitted from this point is shifted to infinitely long wavelength as it propagates out to infinite r.
- Time passes infinitely slowly (relative to a far away observer) at this point (like traveling at $v = c$ in SR).
- Radial velocity $\frac{\Delta r}{\Delta t} = cA(r)$ for light goes to zero there.

More ...

- Notice something else for $r < r_S$ role of time and space coordinates is interchanged – the singularity $r = 0$ is in the "future" of any test particle in this region.
- Thus the event horizon marks a boundary in spacetime. Particles outside this may escape to infinity. Those within it even light are trapped and will eventually reach the singularity.
- Caveat. Schwarzschild only applies to exterior of mass distribution. Thus bodies must be (very) dense for $r_S > \mathrm{physical radius}$. Only then does body have event horizon.
- What is r_S for Sun ?

Embedding diagrams

- Can draw a picture which allows this stretching of space to be visualized. Fix far-way time t The (spatial) curvature can be visualized by *embedding* the surface in a flat 3D space. This *extra* dimension is an aid to visualization only. Resulting picture is called an *embedding diagram*
- The profile of the surface is given by function $A(r)$.
- Clear now why the distance between two points at different r-coordinate is bigger than than the mere difference in r.
- The event horizon corresponds to the point where the slope of the profile is infinite. (this representation does not work for points inside the event horizon).
- Helps make it clear that once inside the event horizon escape is impossible.

More on event horizon

- Have seen that radial velocity of light goes to zero as $r \to r_S$. It then reverses! Light falls towards the singlarity $r = 0$. (why called singularity – because curvature $R \to \infty$ there)
- Spacetime is so curved that the light cones of SR tip over and even light cannot escape!
- Event horizon divides spacetime into 2 causally disconnected regions – nothing from inside the event horizon can influence what happens outside ...

More on FOR

- Notice that an observer close to r_S will not notice anything odd – velocity of a radially moving light beam will be c just as per normal.
- He/she will see no change as the light ray crosses the event horizon, no violent redshifting (to him/her) etc
- Locally all will be well. It is only globally (as seen from the far away FOR (r, t)) that it is obvious a threshold has been crossed ...
- Only one physical motion. Many FOR of reference can be used to view it eg shell, FFF, far away (global) frame. They agree only on invariant intervals not things like distance, times, velocities etc.
- Different systems may be better/worse for figuring out different things eg presence of event horizon.