PHY312 - lecture 13

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Summary so far ...

- (Tidal) gravity caused by geodesic motion on curved spacetime.
- Curved space(time) described by metric - matrix which varies from point to point.
- If spacetime flat possible to choose FFF where

  $$g_{11} = 1, g_{22} = g_{33} = g_{44} = -1$$

  Laws of SR hold. Minkowski space.
- In general curvature R non-zero. So such frames exist only locally.
- Geometrical picture automatically contains POGR, POE

Field equations

- Need equation that determines how curvature R is determined by mass-energy-momentum (density) T.
- Schematically

$$R = \text{constant}\ T$$

where $\text{constant} = G/c^4$

- For small v/c and small curvature reduces to Newtonian gravity.
- What about geodesic motion ?

Geodesic motion

- Conservation of energy-momentum implies that $\frac{\Delta x_i}{\Delta \tau} = \text{constant}$
- Or $\frac{d^2 x_i}{d\tau^2} = 0$ in inertial FOR.
- Consider motion in some arbitrary FOR $y = y(x)$.

$$\frac{dx^i}{d\tau} = \sum_j \frac{\partial x^i}{\partial y^j} \frac{dy^j}{d\tau}$$

and

$$\frac{d^2 x^i}{d\tau^2} = \sum_j \frac{\partial x^i}{\partial y^j} \frac{d^2 y^j}{d\tau^2} + \frac{\partial^2 x^i}{\partial y_j \partial y_k} \frac{dy^j}{d\tau} \frac{dy^k}{d\tau}$$

Finally

Thus free motion looks in funny FOR looks like

$$\frac{d^2y^j}{d\tau^2} + \Gamma^i_{jk}\frac{dy^j}{d\tau}\frac{dy^k}{d\tau} = 0$$

- By the principle of equivalence this equation should also hold for geodesic motion in a curved spacetime (since inertial effects are equivalent to gravitational).
- The difference between the gravitational situation and the flat space free motion is that there will not exist any (global) coordinate system in which all the Γ^s vanish.

Newtonian limit again

- Limit of small v/c expect only Γ^i_{11} to be important.
- Equation of motion looks like

$$\frac{d^2x^i}{dt^2} = -\Gamma^i_{11}c^2$$

- Corresponds to Newton's second law if
  - $\Gamma^i_{11} = \text{force} = -\frac{dU}{dr}$
  - $g_{11} \sim 1 + \frac{U}{c^2}$

Conclusions

Thus Einstein's field equations and the geodesic rule for motion in a curved spacetime indeed contain Newton's law of gravity plus his famous second law of motion in the limit when $v << c$ and gravity is weak

Approx solutions

- Einstein was able to show that GR contained Newton's theory plus small corrections
- Was able to compute these corrections and correctly reproduced
  - Bending of light (verified by Eddington in solar eclipse of 1919).
  - Slowing of clocks
  - Perihelion precession of Mercury. Discrepancy 42 seconds of arc per century.

Schwarzschild solution

- Exact solutions rare and hard to find. Numerical solutions challenging (eg binary black hole collision project, needs supercomputer level effort)
- However in some very simple cases possible - eg Schwarzschild solution. First exact solution to GR. Describes spacetime outside spherically symmetry, static mass distribution. 1915.
- Applies to Sun, Earth, neutron star, black hole, ...

Metric

- First lets write spacetime of (2+1) SR in polar coordinates (r, θ).

$$\Delta s^2 = c^2 \Delta t^2 - \Delta x^2 - \Delta y^2$$

becomes

$$\Delta s^2 = c^2 \Delta t^2 - \Delta r^2 - r^2 \Delta\theta^2$$

- Note: spacetime is still flat but metric is now non-trivial in polar coordinates ...
- $2\pi r$ is distance round spherical object at r coordinate r.
- Switch on gravity. How will this change ?
- Spherical symmetry (and time independence) suggests we try

$$\Delta s^2 = A(r) c^2 t^2 - B(r) \Delta r^2 - r^2 \Delta\theta^2$$

Solution

- Plug into field equations. Find solution provided
  - $B(r) = 1/A(r)$
  - $A(r) = 1 - \frac{2GM}{c^2 r}$
- Dimensions ?
- Time t is far away time. Measured by clock at infinity.

Comments

- Einstein wrote to Karl Schwarzschild “I had not expected that the exact solution to the problem could be formulated. Your analytic treatment seems to me splendid”.
- This is valid for *timelike* separated events. For spacelike events multiply by minus 1.
- This is nonrotating, uncharged, spherically symmetric structure. The solution for a spinning black hole was only published in 1963 almost 50 years later! Holds all info on the external spacetime.
- Note: $r \to \infty$ corrections go to one - flat spacetime. Also for $M \to 0$ - correct - flat spacetime again.