PHY312 - lecture 12

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Recap

- Ingredients of GR – principle of equivalence, principle of general relativity. Importance of tidal gravity.
- Consequences of POE – slowing of clocks in gravitational field, deforming of spatial geometry, bending of light ...
- Important hints – perhaps (tidal) gravitational force corresponds to free motion in curved spacetime
- Simplest example of curved space – surface of sphere. Geodesics. Curvature - intrinsic definition.

Curved spaces

- Curved spaces described using a metric g_{ij} Distance between two nearby points $x_i \equiv (x_1, x_2, \ldots)$ and $x_i + \Delta x_i \equiv (x_1 + \Delta x_1, \ldots)$ is

$$\Delta s^2 = \sum_i \sum_j g_{ij} \Delta x_i^2 \Delta x_j^2$$

- Using g_{ij} can compute
  - Geodesics
  - Curvature
  - Any other geometrical feature of space.
- All these things independent of which coordinate system used. Note: locally – any curved space looks flat.

Curved spacetime

- Proceed by analogy $x_\mu \equiv (ct, x, y, z)$. Spacetime coordinate. Coordinate system replaced by FOR.
- Distance in spacetime (squared) given by

$$ds^2 = \sum_{\mu=1}^{4} \sum_{\nu=1}^{4} g_{\mu\nu} dx^\mu dx^\nu$$

- Curvature defined as for space. Also geodesics.
- Physical quantities eg curvature independent of FOR.
- Locally any curved spacetime looks flat – can find FOR (FFF) locally where metric of special relativity valid .. POE

Comparison

- Geometry of curved spaces independent of coordinates – geometry of curved spacetime independent of FOR or Principle of General Relativity.
- Geodesics - closest thing to straight lines in curved space and minimize distance between 2 pts – similarly in spacetime but maximises spacetime distance (proper time)
- If curvature of space R vanishes can choose coordinates (Cartesian) where

  $$g_{11} = g_{22} = \ldots = 1$$

  – if curvature of spacetime vanishes – can choose FOR (inertial)

  $$g_{11} = c, g_{22} = -1, g_{33} = -1, g_{44} = -1$$

GR

- Einstein replaces motion under gravity with geodesic motion on curved spacetime.
- But why is spacetime curved ? To correspond with Newtonian gravity it must curve in response to presence of massive bodies like Sun.
- But mass is not good concept in SR. Energy-momentum better.
- Schematically Einstein replaces Newtonian inverse square law with field equations

$$\mathrm{Curvature\ R = constant Density\ of\ energy - momentum\ T}$$

The constant here must involve Newton's gravitational constant G_N.

Newtonian limit

- The idea that gravity is geometry naturally satisfies POE, POGR and ideas about tidal gravity.
- And clearly the source must be something like energy-momentum $\mathcal{P}$.
- But there are many field equations of previous general form that would embody all this. What fixes the true equations ?
- Ans: the requirement that they reduce to Newton’s theory in limit
  - $v/c \rightarrow 0$
  - (Tidal) gravity is weak

Newtonian gravity recast

- Introduce Newtonian gravitational potential energy $U = -GM/r$ which gives energy per unit mass in field of large mass M.
- In general force $F = -\frac{dU}{dr}$
- Tidal forces $F_T = -\frac{dF}{dr} = -\frac{d^2U}{dr^2} \sim GM/r^3$
- Newton's inverse square law rephrased as just

$$\frac{d^2U}{dr^2} = G\rho$$

  where ρ is mass density.
- Thus this equation should fall out of field equations for v/c small and weak tidal gravity.

More handwaving

- How is curvature related to metric ? For weak gravity expect spacetime roughly like SR (Minkowski).
- Thus g_{11} larger than g_{22} etc by factor c^2 and dominates R.
- Dimensions of curvature $1/L^2$ (see discussion of sphere). So expect roughly $R \sim \frac{d^2 g_{11}}{dr^2}$
- Energy-momentum T_{11} should be dominated by energy density ρc^2
- This Einstein's equation should reduce to

$$\frac{d^2 g_{11}}{dr^2} = \text{constant } \rho c^2$$

Baisc idea

- Implies in weak field that $1 + \frac{U}{c^2} = g_{11}$
- Constant in EFE $\sim G/c^4$
- Further constraints ?

Geodesic motion

- Conservation of energy-momentum implies that $\frac{\Delta x_i}{\Delta \tau} = \mathrm{constant}$
- Or $\frac{d^2 x_i}{d\tau^2} = 0$ in inertial FOR.
- Consider motion in some arbitrary FOR $y = y(x)$.

$$\frac{dx^i}{d\tau} = \sum_j \frac{\partial x^i}{\partial y^j} \frac{dy^j}{d\tau}$$

and

$$\frac{d^2 x^i}{d\tau^2} = \sum_j \frac{\partial x^i}{\partial y^j} \frac{d^2 y^j}{d\tau^2} + \frac{\partial^2 x^i}{\partial y_j \partial y_k} \frac{dy^j}{d\tau} \frac{dy^k}{d\tau}$$

Finally

Thus free motion looks in funny FOR looks like

$$\frac{d^2y^j}{d\tau^2} + \Gamma^i_{jk}\frac{dy^j}{d\tau}\frac{dy^k}{d\tau} = 0$$

- By the principle of equivalence this equation should also hold for geodesic motion in a curved spacetime (since inertial effects are equivalent to gravitational).
- The difference between the gravitational situation and the flat space free motion is that there will not exist any (global) coordinate system in which all the Γ^s vanish.

Newtonian limit again

- Limit of small v/c expect only Γ^i_{11} to be important.
- Equation of motion looks like

$$\frac{d^2x^i}{dt^2} = -\Gamma^i_{11}c^2$$

- Corresponds to Newton’s second law if
  - $\Gamma^i_{11} = \text{force} = -\frac{dU}{dr}$
  - $g_{11} \sim 1 + \frac{U}{c^2}$

Conclusions

Thus Einstein’s field equations and the geodesic rule for motion in a curved spacetime indeed contain Newton’s law of gravity plus his famous second law of motion in the limit when $v << c$ and gravity is weak