

# PHY312 - lecture 11

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# Recap

- Ingredients of GR – principle of equivalence, principle of general relativity. Importance of tidal gravity.
- Consequences of POE – slowing of clocks in gravitational field, deforming of spatial geometry, bending of light ...
- Important hints – perhaps (tidal) gravitational force corresponds to **free motion** in curved spacetime
- Simplest example of curved space – surface of sphere. Geodesics.

# Curved Space

- Simplest example of **non-Euclidean** geometry.
- Study it need to employ a coordinate system eg.  $(x, y, z)$  with

$$x^2 + y^2 + z^2 = R^2$$

- Or spherical polar coordinates  $\{r, \theta, \phi\}$  with  $r = R$ .
- To talk about geometry we need to talk about distances between (nearby) points  $\{\theta, \phi\}$  and  $\{\theta + \Delta\theta, \phi + \Delta\phi\}$ .
- If I know all these distances I can reconstruct the surface ...

$$\Delta s^2 = R^2 (\Delta\theta^2 + \sin^2 \theta \Delta\phi^2)$$

# Metric

- Can rewrite in a suggestive way ...

$$\Delta s^2 = g_{11}(\Delta x_1)^2 + g_{22}(\Delta x_2)^2$$

where  $x_1 = \theta$ ,  $x_2 = \phi$  and

$$g_{11} = R^2$$

$$g_{22} = R^2 \sin^2 \theta$$

- $g_{11}$  and  $g_{22}$  are components of an object called the **metric**
- Knowledge of the metric at each point **uniquely** determines the physical surface (geometry).
- Flat space  $g_{11} = g_{22} = 1$ . For curved geometry metric will vary from point to point ....

# Lengths

- If we know metric can compute distances between 2 points which are not close together ...
- Consider the path  $\phi = \text{constant}$  varying  $\theta$ .

$$L = R \int_0^\pi d\theta = \pi R$$

Example of a **geodesic**. Shortest path between two points. Look locally like straight lines ...

- Set of paths with  $\theta = \text{constant}$  varying  $\phi$ .

$$L = R \sin \theta \int_0^{2\pi} d\phi = 2\pi R \sin \theta$$

Not a geodesic.

# Geodesics

- Any truly **curved** surface possesses neighboring geodesics which do not remain parallel over whole surface.

# Curvature

- How can one tell locally that the surface is not flat ?  
Ans: compute the **curvature**.
- Take a small vector arrow and move it around a closed path on the surface so that it always makes a fixed angle with the tangent vector to the curve at that point (this is called **parallel transport**). Compare the final vector with the initial vector and determine the angle of rotation that has been induced by the motion. Then the curvature is given by the formula

$$R = (\text{net rotation angle}) / (\text{surface area enclosed by loop})$$

# Curvature of sphere

- Take path that runs from equator to N pole. Then along similar path back to equator arriving 1/4 way along equator from start.
- What is curvature of sphere ?
- Thus we don't need to "look at the sphere from a higher dimensional space to see it is curved". We can determine that property by just moving on loops on the sphere and measuring angles ..
- For general curved space these loops must be small and the curvature varies from point to point ...
- What happens for  $R \rightarrow \infty$  and the sphere approaches a flat plane ?

# General coordinate systems

- One final complication. Want to describe sphere using any coordinate system. Not just one whose basis vectors are at  $90^\circ$  to each other.
- Eg even in flat space we might want to transform from Cartesian system  $(x, y)$  to some other system  $(x', y')$ .

$$x' = ax + by$$

$$y' = cx + dy$$

- Original distance  $l^2 = (x')^2 + (y')^2$  is now given by

$$l^2 = (a^2 + c^2)x^2 + (b^2 + d^2)y^2 + 2(ab + cd)xy$$

- Thus need off-diagonal components of metric  $g_{12}$  and  $g_{21}$ .

# General formula for length

- $L^2 = \sum_{i=1}^2 \sum_{j=1}^2 g_{ij} x_i x_j$
- Metric is a **matrix** at each point of space.
- Similarly curvature also has several components and varies over surface (**tensor**).
- Mathematical description gets involved - tensor calculus and we won't get into to it here

# Curved spacetime

- Proceed by analogy  $x^\mu = (t, x, y, z)$ . Spacetime coordinate. Coordinate system replaced by frame of reference.
- Distance in spacetime (squared) given by differences in nearby spacetime coordinates weighted by metric.

$$ds^2 = \sum_{\mu=1}^4 \sum_{\nu=1}^4 g_{\mu\nu} dx^\mu dx^\nu$$

- Question: What is the metric of special relativity ?
- Locally can always use a frame of reference (coordinate system) such that  $g_{11} = c, g_{22} = -1, g_{33} = -1, g_{44} = -1$  and all other components zero – FFF. Principle of equivalence.

# Curvature II

- Mathematical expression exists for curvature in spacetime analogous to the curvature of space.
- If  $R_{ne0}$  **cannot** find a FFF frame where metric is simple everywhere (presence of tidal gravity)

# Motion

- In SR we learned that free motion corresponds to straight worldlines and a **maximal** proper time. It is hence geodesic motion in a flat spacetime.
- In GR we postulate that free motion corresponds also to maximal proper time but now in general the geodesic worldlines are **curved**.
- In this way we remove the force of gravity and replace it by geodesic motion in curved spacetime.
- Notice this structure automatically encompasses the principle of general relativity since in such a geometric theory – the curved space(time) is real – any coordinate system/ FOR is equally good to describe it.

# To do ...

- Need to flesh out this geodesic motion.
- Need to give an equation which tells us how curved is spacetime ... This must have something to do with the distribution of matter and energy ... Einstein's **field equations**
- Show that these reduce to Newton's theory for small velocities and weak gravity ...