Solutions to Homework Assignment #9 – PHY312

8-2. A) OK, for each value of r, we want to consider the corresponding circle about the center and to compute its radius R and its circumference C. Actually, computing the radius is easy. As we found in class, when the coefficient (g_{rr}) in front of the dr^2 term is 1, the radius is just $R = r$.

i) So, now we want to compute the circumference of the circle at r for the metric $ds^2 = dr^2 + a^2 sin^2(r/a)d\theta^2$. Since r will remain constant around the circle, the change in r (dr) is zero: $dr = 0$. Thus, along the circle we have

$$ds^2 = a^2 \sin^2(r/a)d\theta^2,$$

or,

$$ds = a\sin(r/a)d\theta.$$

The circumference is the length of this circle:

$$C = \int ds = \int_0^{2\pi} a\sin(2\pi r/a)d\theta = (2\pi)a\sin(r/a).$$

So, $C/R = (2\pi)(a/r)\sin(r/a)$. Let's define $x = a/r$ so that $C/R = (2\pi)\frac{\sin x}{x}$. Now, $\sin x < x$, so $C/R < 2\pi$. Note that C/R does become nicely 2π in the limit as $x \to 0$ (since $\sin x \approx x$ for small x); that is, if the circle is small enough to be 'local.'

ii) Once again, $dr = 0$ for our circle. Much as before, we find $ds = a\sinh(a/r)d\theta$, so the circumference is

$$C = \int ds = \int_0^{2\pi} a\sinh(2\pi r/a)d\theta = (2\pi)a\sinh(r/a).$$

Now, $C/R = (2\pi)(a/r)\sinh(r/a)$. Again, let's define $x = a/r$. Then, $C/R = (2\pi)\frac{\sinh x}{x}$. Now, $\sinh x > x$, so $C/R > 2\pi$. Note that C/R once again nicely becomes 2π in the limit as $x \to 0$ (since $\sinh x \approx x$ for small x).

B) For this part, the coefficient ($g_{\theta\theta}$) of the $d\theta^2$ term is just ρ^2. For any circle around the origin, $d\rho = 0$ so $C = \int\ ds = \int_0^{2\pi} \rho\ d\theta = 2\pi\rho$. Thus, the interesting question is: How long is the radius?

Let's compute the actual length of the radius for each case. Along a radial line, $d\theta = 0$.

a) So, for the first metric, the length R of the radial line from 0 to ρ is

$$R = \int ds = \int_0^\rho \frac{a}{\sqrt{a^2 - \rho^2}} d\rho = a \sin^{-1}(\rho/a).$$

If you're not familiar with this integral, you can work it out using a substitution like $\rho = a \sin\theta$.

In other words, $\rho = a\sin(R/a)$. Thus, $C = 2\pi a \sin(R/a)$. Does this look familiar? It is the same relationship we found for metric (i). In fact, metric (i) and metric (a) represent the same *geometry*, they just use different coordinates. Can you see what the relationship is between r and ρ? As you might be able to tell, these are both metrics for a sphere of radius a.

b) For the second metric, the length R of the radial line from 0 to ρ is

$$R = \int ds = \int_0^\rho \frac{a}{\sqrt{a^2 + \rho^2}} d\rho = a \sinh^{-1}(\rho/a).$$

If you're not familiar with this integral, you can work it out using a substitution like $\rho = a \sinh\theta$.

In other words, $\rho = a\sinh(R/a)$. Thus, $C = 2\pi a \sinh(R/a)$. This is exactly the relationship we found for metric (ii). It turns out that metrics (ii) and (b) once again represent the same geometry using different coordinates. Can you see what the relationship is between r and ρ in this case? As you might be able to tell, these are both metrics for a Lobachevskian space.

c) For metric (i), we found $\rho = a\sin(R/a)$, so $d\rho = a\cos(R/a)dR$. Now, we just substitute:

$$\begin{aligned} ds^2 &= \frac{d\rho^2}{a^2 - \rho^2} + \rho^2 d\theta^2 \\ &= \frac{a^2\cos^2(R/a)dR^2}{a^2\left(1 - \sin^2(R/a)\right)} + a^2\sin^2(R/a)d\theta^2 \\ &= \frac{a^2\cos^2(R/a)dR^2}{a^2\cos^2(R/a)} + a^2\sin^2(R/a)d\theta^2 \\ &= dR^2 + a^2\sin^2(R/a)d\theta^2. \end{aligned} \quad (1)$$

So, this is just the same as metric (a), for which we had $R = r$!

Similary, for metric (ii), we found $\rho = a\sinh(R/a)$, so $d\rho = a\cosh(R/a)dR$. Now, we just substitute:

$$ds^2 = \frac{d\rho^2}{a^2 + \rho^2} + \rho^2 d\theta^2$$

$$
\begin{aligned}
&= \frac{a^2\cosh^2(R/a)dR^2}{a^2\left(1+\sinh^2(R/a)\right)} + a^2\sinh^2(R/a)d\theta^2 \\
&= \frac{a^2\cosh^2(R/a)dR^2}{a^2\cosh^2(R/a)} + a^2\sinh^2(R/a)d\theta^2 \\
&= dR^2 + a^2\sinh^2(R/a)d\theta^2. \qquad (2)
\end{aligned}
$$

This is the same as metric (b), for which we also had $R = r$.

9-2. Around one of these circles, t, r, and θ are constant. So, we have $dt = dr = d\theta = 0$. Since $\theta = \pi/2$ and $\sin(\pi/2) = 1$, we have $ds^2 = r^2 d\phi^2$, or $ds = rd\phi$. Thus, the circumference of such a circle is:

$$C = \int_0^{2\pi} r d\phi = 2\pi r.$$

Thus, $dC/dr = 2\pi$. But what is dC/dR?? To compute this, we must find dr/dR. So, let's consider a line that goes 'straight out' in the radial direction in some constant time slice. This line will have $d\theta = d\phi = dt = 0$. The actual Radius R is just the *distance* along this line. So, $dR = ds = dr/(1-R_s/r)^{1/2}$.

Using the chain rule, we find

$$a) \qquad \frac{dC}{dR} = \frac{dr}{dR}\frac{dC}{dr} = 2\pi(1 - R_s/r)^{1/2} < 2\pi.$$

b) We see that the circumference increases more slowly than it would in flat space. When r is close to R_s (i.e., near the horizon of a black hole), this effect is very large and dC/dR is almost zero. A detailed picture of this is shown at the bottom of page 245 in the Notes. For reasons that we will discuss later, $r = R_2$ is the circle at the 'neck' in the middle of the diagram.

Look at this circle ($r = R_s$) and compare it to the next circle farther 'out' in either direction. The next circle is only slightly bigger than the one at $r = R_s$, so dC/dR is nearly zero.

On the other hand, when we are far from the (potential) horizon and $r \gg R_s$, R_s/r is almost zero and dC/dR is essentially 2π. This means that the space is very close to flat, as shown in the picture.