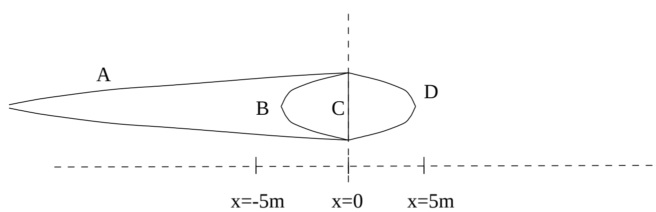
Solutions to Homework Assignment #8 – PHY312

7-3. The diagram on the homework assignment was drawn in the reference frame of a lab sitting on the earth. As we have now learned, the best reference frame in which to draw such a diagram is a freely falling frame. Then, things work (at least locally) just like they do in inertial frames. So, what will this diagram look like in a freely falling reference frame?

Well, let's think about each of the worldlines drawn on the diagram. Worldline A first goes down and then comes back up. On the other hand, both C and D first go up and then come back down. So, both C and D look something like the worldlines of objects that are just flying through the air .... in other words, they looks something like the worldlines of objects in *free-fall.* Are they really the worldlines of freely falling objects? Worldline C goes up 5m and falls back down in two seconds. However, worldline D goes up about 50m and falls back down in two seconds. If I take some object (like my favorite eraser) and toss it 5m up into the air [try this yourself], I see from my little experiment that it takes roughly two seconds to come back down. If I could throw it up 50m, it would take much longer to fall back down. So, I conclude that worldline C is pretty much a freely falling object, while worldline D is not.

OK then, I'd like to redraw this diagram in the reference frame of a freely falling object. So, why not use the reference frame of C? In that case, it is easy to draw C's worldline (a straight vertical line) while worldlines A and B curve below it and worldline D curves above it. The result looks like:



and we can interpret this just as we would a spacetime diagram in an inertial reference frame.

So, which worldline represents more proper time? Since our frame of reference is effectively inertial, the answer is 'the straightest one.' This is worldline C, the worldline of the freely falling object.

7-4. [optional] As shown in the notes, if the gravitational field is g, and two clocks (A and B) are at heights r_A and r_B above the center of the earth, the rate at which time t_A accumulates on clock A is related to the rate at which time t_B accumulates on clock B by

$$\frac{t_A}{t_B} = \exp(\int_{r_B}^{r_A} dr\ g/c^2).$$

Now,

$$\int_{r_B}^{r_A} dr\ (GM/r^2 + a/r^3 + b/r^4) = (-GM/r - a/2r^2 - b/3r^3)|_{r_B}^{r_A}.$$

So,

$$\frac{t_A}{t_B} = \exp[\frac{GM}{c^2}(r_B^{-1} - r_A^{-1}) + \frac{a}{2c^2}(r_B^{-2} - r_A^{-2}) + \frac{b}{3c^2}(r_B^{-3} - r_A^{-3})].$$

7-5: Here I have in mind that we use the formula

$$\frac{t_{Denver}}{t_{DC}} = \exp[\frac{GM}{c^2}(r_{DC}^{-1} - r_{Denver}^{-1})].$$

from section 7.4. We have:

$$\begin{aligned} G &= 6 \times 10^{-11} Nm^2/kg^2, \\ m_E &= 6 \times 10^{24} kg, \\ r_{DC} &= r_0 = 6 \times 10^6 m, \\ r_{Denver} &= r_0 + 1600m = 6.0016 \times 10^6 m. \end{aligned}$$

As a result,

$$\frac{t_{Denver}}{t_{DC}} = \exp[1.8 \times 10^{-13}] \approx 1 + 1.8 \times 10^{-13}.$$

7-6: [optional] Let's estimate the speed at which DC and Denver move around the earth. The earth rotates once every 24 hours, so roughly speaking, DC moves at $v_{DC} \sim 2\pi r_{DC}/24hrs = \frac{12\pi \times 10^6 m}{86{,}400 sec} = 436 m/s$, while Denver moves at $v_{Denver} \sim 2\pi r_{Denver}/24hrs = \frac{12.0032\pi \times 10^6 m}{86{,}400 sec} = 436 m/s$.

Let's calculate a time dilation factor for both relative to the center of the earth. For DC, $\sqrt{1 - v_{DC}^2/c^2} \approx 1 - \frac{1}{2}\frac{v_{DC}^2}{c^2} = 1 - 2.1134 \times 10^{-12}$. For Denver, $\sqrt{1 - v_{Denver}^2/c^2} \approx 1 - \frac{1}{2}\frac{v_{Denver}^2}{c^2} = 1 - 2.1144 \times 10^{-12}$. So, ratio is

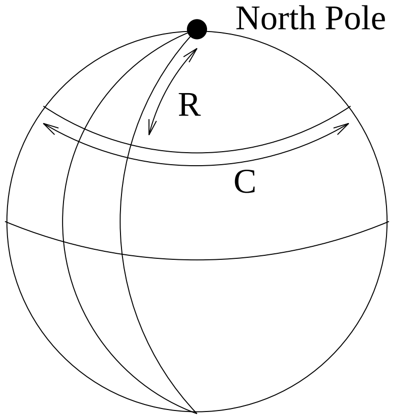
$$\frac{\sqrt{1 - v_{DC}^2/c^2}}{\sqrt{1 - v_{Denver}^2/c^2}} = 1 + 1.1 \times 10^{-15}.$$

This is much smaller than the effect computed in 7-5 and can be ignored. However, it turns out that it does become important for GPS satellites.

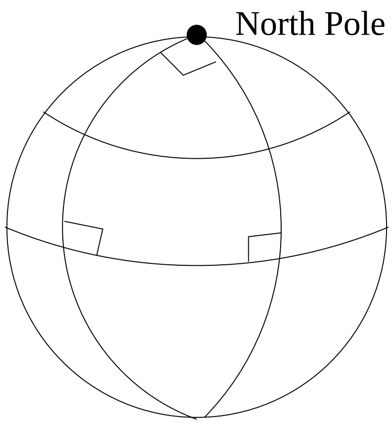
8-1. This problem mostly involves doing the kinds of things that we talked about in class – looking at geodesics, circles, triangles, and at what happens when you move arrows (vectors) around closed loops. I think it will be easiest if I organize things by the shapes, first doing (a) through (d) for the sphere, then for the cylinder, etc.

i) The Sphere: We actually did most of this in class (see section 8.2 of the notes), but let me just repeat things briefly.

a) Let's look at a circle around the north pole. Think about taking a metal ring of some given size (circumference) and fitting it over the north pole of a sphere. Note that the radius (as measured *along* the sphere) is bigger than the radius would be in flat space. This is because the radius has to curve up to reach the north pole. Thus, for a given circumference C, the radius is bigger than it would be in flat space. In other words, $R > C/2\pi$, or $C < 2\pi R$.

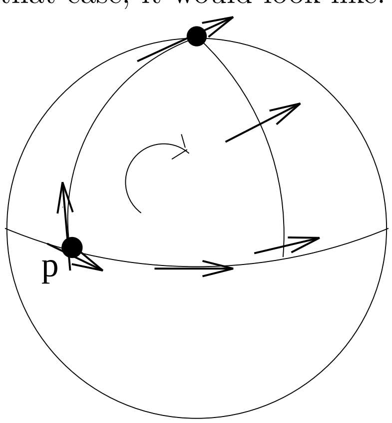


b) I can draw a triangle on a sphere whose angles add up to 270 degrees. Each of the angles shown below is a right angle:



c) From the above picture, it is clear that initially parallel geodesics bend *toward* each other. You can see this by considering the two legs of my triangle that run from the equator up to the north pole. They start off parallel, but then they come together.

d) Let's try carrying an arrow around this triangle. Suppose that we start at point p with an arrow pointing north. We then walk around the triangle (clockwise, as shown below) and carry the arrow with us *without rotating it*. In that case, it would look like:

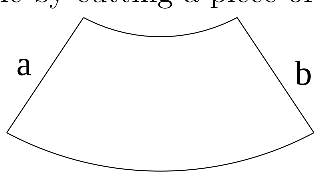


We can see that, when we return to point p, the arrow is now pointing straight east! Thus, it has been rotated 90 degrees clockwise – the same direction as we went around the triangle.

ii) The Cylinder: The best way to do this one is experimentally – that is, to actually make a cylinder out of paper, get out a pencil, a thumbtack, and some string, and give it a try! What you will find is rather interesting and, it retrospect, you can probably visualize it. The point is this: suppose you make a paper cylinder by rolling up a piece of paper. If you draw things on your cylinder and then unroll the cylinder into a flat piece of paper, you don't change the length of any lines, and you don't change any angles! As a result, *everything you draw on a cylinder comes out just the same as it would on a flat piece of paper!*. Circles satisfy $C = 2\pi R$, triangles have 180 degrees, initially parallel geodesics remain parallel, and arrows don't rotate when you 'parallel transport' them around closed loops!

OK, now why is this the right way to think about things? Recall that the geometry of a surface is entirely determined by the distances between pairs of points... i.e., by the distances between molecules in the paper. But these distances are determined by the molecular bonds and do *cannot* change when we bend the paper (unless we cut the paper and remove some of the molecules). So, these distances (and thus the geometry) remain the same when we roll the paper up into a cylinder.

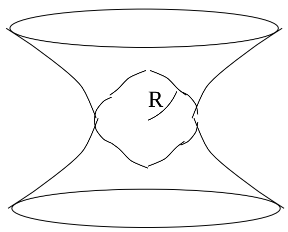
iii) The Cone: The best way to do this one is experimentally – that is, actually make a cone out of paper, get out a pencil, a thumbtack, and some string, and give it a try! What you will find is rather interesting and, it retrospect, you can probably visualize it. The point is this: suppose you make a paper cone by cutting a piece of paper like this:



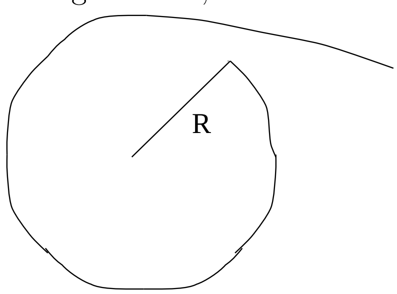
and then rolling it up by taping together sides (a) and (b). If you draw things on your cone and then unroll the cone into a flat piece of paper, you don't change the length of any lines, and you don't change any angles! As a result, *everything you draw on a cone comes out just the same as it would on a flat piece of paper!*. Circles satisfy $C = 2\pi R$, triangles have 180 degrees, initially parallel geodesics remain parallel, and arrows don't rotate when you 'parallel transport' them around closed loops!

iv) The Double Funnel: Once again, it may be best to find a funnel somewhere and try this 'experimentally.' But, I'll try to draw the right pictures below.

a) A circle drawn on the double funnel looks roughly like:

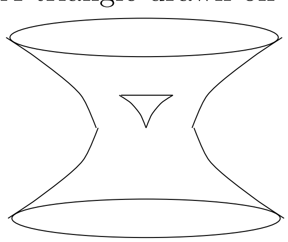


Let's think about what would happen if you tried to smash the circle down into flat space. All of those wrinkle would have to be ironed out. Imagine taking an iron and, starting at one point on the circle, smashing it down. This will make the circle spread out more and more. After you finish smashing it down, it would look something like:



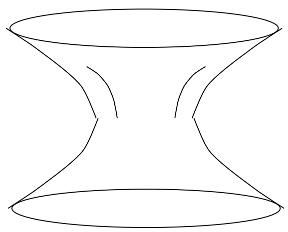
So, the circumference is too big for the radius: $C > 2\pi R$.

b) A triangle drawn on the Double Funnel looks like:

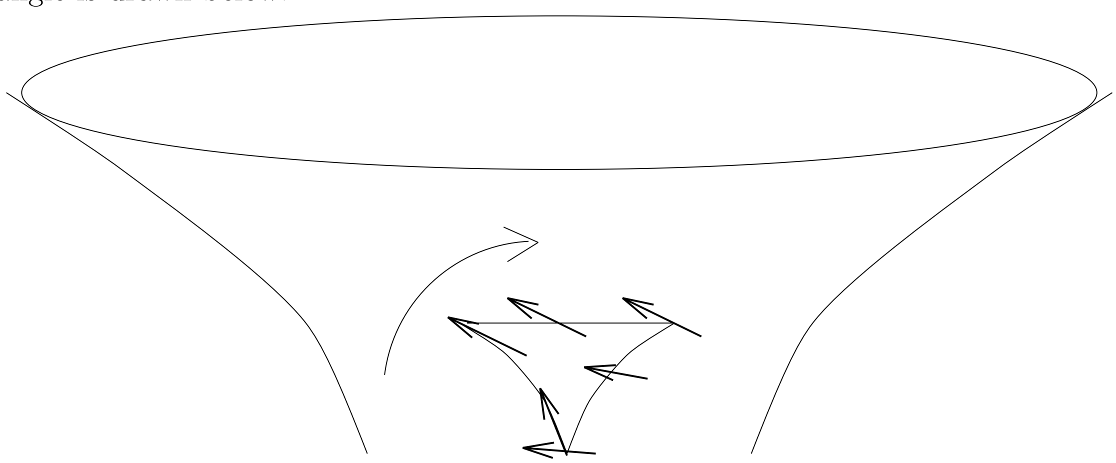


As you can see, each of the angles in this triangle is less than 60 degrees, so the sum of the angles is less than 180 degrees. [Note that if you connect the 3 corners above with lines that are "straight" according to this flat piece of paper, you draw a new triangle that is *outside* the old one – i.e., that has *bigger* angles.]

c) Geodesics drawn on the Double Funnel look like:

 As you can see, they start off parallel at the bottom, but then they move apart as they go up the funnel.

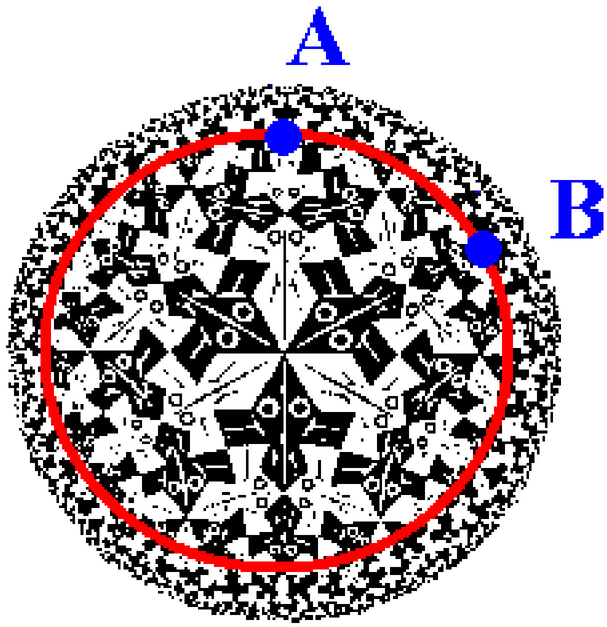
d) Let's go back to our triangle above and, starting at the bottom corner, carry an arrow around it. As before, we will start out with an arrow pointing straight 'ahead' along the first side. We will walk around the path clockwise, as indicated below. Note that the arrow we carry will still be pointing straight ahead when we get to the second corner. The complete trip around the triangle is drawn below:



We can see that, along the second side, the arrow we carry points about 150 degrees to the left of straight ahead. Finally, along the third side, it points about 60 degrees to the right of straight ahead. When we get back to the bottom corner, the arrow has been rotated 60 degrees *counterclockwise* relative to where it started. In other words, it has rotated in the direction *opposite* to that in which we went around the triangle.

v) Finally, it's time to work with the Lobachevskian space.

a) Let's start by drawing a circle on the Lobachevskian space. I'll put the center of my circle right in the middle of the picture. For convenience, I'll make the radius of the circle an integer number of fish, say, $R = 2\ Fish$. The way that I drew this circle was to start in the middle, and then count out two fish along each of the three black fish at the center and mark a dot with my pen. Then I did the same along each of the three white fish at the center. I then connected the dots to make a circle:



Escher's depiction of Lobachevskian space
(Think of all the black fish as congruent
and all the white fish as congruent.)

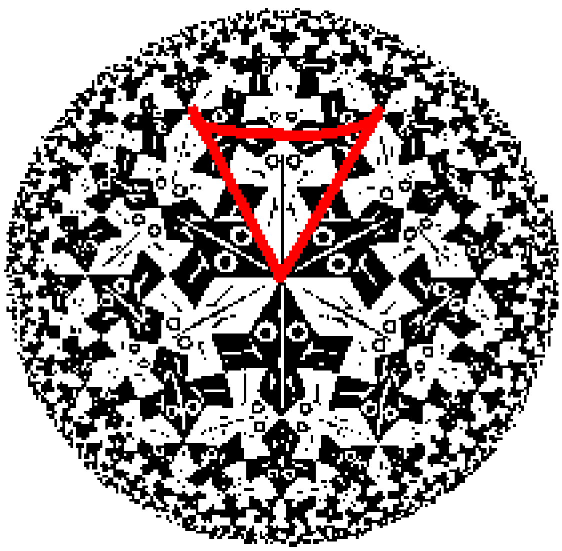
So, this is a circle with $R = 2\ Fish$. But what is the circumference? Let's start by figuring out how long the curve is that connects dots A and B. The circumference of the whole circle is then six times as long as the curve (since that curve goes one sixth of the way around the circle). Starting at A and tracing the curve toward B, the curve goes along most of a white fish, and then most of a black fish. Let's count that as two fish lengths so far. Then, it cuts across little bits of a white fish and black fish. Those bits are so little that let's not count them, so we're still only up to about 2 fish lengths. But then, the curve once again traces most of a white fish and most of a black fish before arriving at B. Therefore, the length of the curve between A and B is roughly $4\ Fish$. The circumference of the circle is then six times as long, roughly $C \cong 24\ Fish$. So,

$$\frac{C}{R} \cong \frac{24\ Fish}{2\ Fish} = 12 > 2\pi.$$

Or, $C > 2\pi R$.

[Note: Depending on how big you made your circle, you may get a different value for C/R. However, you will always find that C/R is bigger than 2π.]

b) This time, we want to draw a triangle. The key point here, of course, is that each side of the triangle should be a *geodesic*. So, how do we know where the geodesics are in this space? The sides of the fish and the lines drawn up their centers form geodesics. So, I can connect the three dots below by following the lines made by the fish as shown:

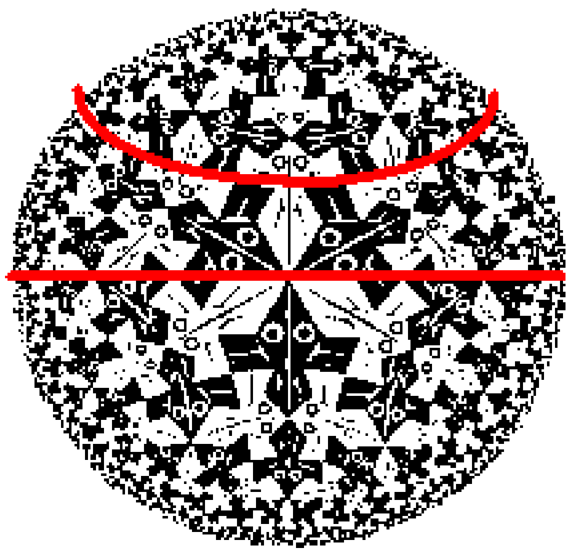


Escher's depiction of Lobachevskian space
(Think of all the black fish as congruent
and all the white fish as congruent.)

Now, what are each of the angles in this triangle? Let's start with the corner in the middle of the diagram. The first angle is one fish wide. Since six fish fit around the central point, an angle one fish wide must be worth $360^o/6 = 60^o$. Now, what about the other two angles? They are each only *half* a fish wide. So, each of them is worth only 30^o!. Thus, the sum of all of the angles in this triangle add up to only 120^o, less than 180^o!

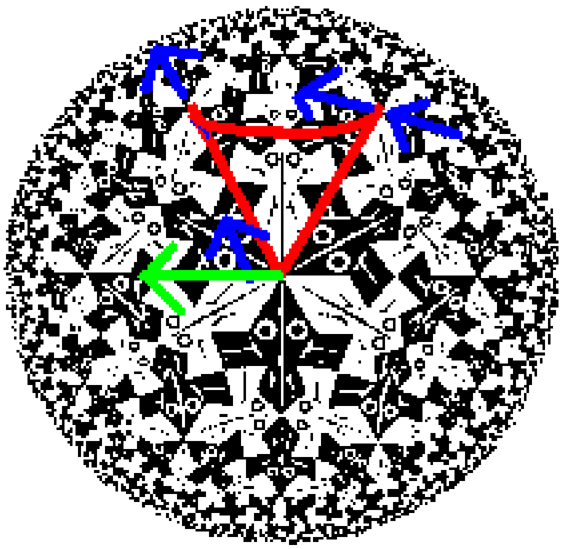
[Note: Depending on how big you made your triangle, you may get a different value for the sum of the angles. However, it will always be *less* than 180^o!]

c) We can use the same technique as above to draw geodesics. I have drawn two of them below. You can see that they start off parallel in the middle of the page, but then they separate. So, initially parallel geodesics bend *away* from each other on the Lobachevskian space.



Escher's depiction of Lobachevskian space
(Think of all the black fish as congruent
and all the white fish as congruent.)

d) Below, I have once again drawn the same triangle as above, and I have drawn what happens to an arrow as I 'parallel transport' it around the triangle. I started in the middle of the picture with an arrow pointing along one side, and I carried the arrow around the triangle in a clockwise direction.



Escher's depiction of Lobachevskian space
(Think of all the black fish as congruent
and all the white fish as congruent.)

I know this is hard to see, so let me describe what is happening. I start in the center of the picture, facing about 30° (half a fish) west of north. This is the smaller arrow at the center – the one which is blue in the color version. My arrow is pointing in the same direction, and I walk along that geodesic. Since the arrow starts out pointing straight ahead, it will point straight ahead until I reach the next corner. I have drawn another small blue arrow here to show this.

I then walk along the next side of the triangle, *without turning the arrow.* Note that the arrow starts off pointing more or less behind me, but also to my left about half a fish. Therefore, since I do not rotate the arrow, when I reach the next corner the arrow will still point mostly behind me, but half a fish to my left.

Now I start walking down the third side of the triangle. The arrow starts out pointing one *whole* fish to the right of straight ahead. So, when I get back to the center of the picture, the arrow still points one whole fish to the right of straight ahead. The final arrow (the large green one at the center) differs from the original arrow by one whole fish (60^0, see above) and has rotated *counterclockwise* – i.e., *opposite* to the direction I went around the loop.

Note that, qualitatively, in every case the results for the Lobachevskian space agreed with the results for the double funnel. These properties are always grouped together, and spaces like these are called *negatively curved* spaces. This differentiates them from *positively curved* spaces like the sphere, which always produced the opposite results.

So, the Lobachevskian space is somewhat like the Double Funnel. You might ask, is it in fact the double funnel, but just drawn in a different way? The answer is no. The difference can been seen in the fact that the Lobachevskian space is *homogeneous*; that is, every black fish in the space is the same as every other (and similarly for the white fish). Thus, every part of the Lobachevskian space looks the same. This is not true for the double funnel. You can see that the double funnel is most strongly curved near the middle and then becomes more and more like a cone (which, as we saw in part (iii) is actually *flat*) as we go up or down from there. It turns out that you cannot draw a picture of the Lobachevskian space as was done for the sphere, cylinder, cone, and double funnel. Technically speaking, the Lobachevskian space cannot be "embedded in" (drawn as a surface inside of) flat three-dimensional space. (See your favorite differential geometry or Non-Euclidean geometry book for a proof.)