Solutions to Homework Assignment #7 – PHY312

This handout also contains the solutions to the extra optional problems I assigned before Spring Break

6-1. Well, let's see. I'm about $100kg$, so my rest mass is $m_0 = 100kg$. That means that I already have an energy of $E = m_0c^2 = (100kg)(3 \times 10^8 m/s)^2 = 10^{19} J$ (since $3^2 = 10$). If I were moving at $.9c$, I would have an energy of $E = m_0c^2/\sqrt{1 - v^2/c^2} = (2.3)m_0c^2$. So, my energy would have to increase by $\Delta E = (1.3)m_0c^2 = 1.3 \times 10^{19} J$. Therefore, this is how much energy it would take to accelerate me up to $.9c$.

6-2. One joule is $\frac{1}{3.6} \times 10^{-6}$ kiloWatt-hours, so the energy above is roughly 3.6×10^{12} kiloWatt-hours. At \$0.107 each, this would cost $\$3.8 \times 10^{11}$, or about 400 billion dollars.

6-3. Here, we are combining three objects:
photon #1: energy = E_0, momentum = E_0/c.
I found the momentum by remembering that, for light, $p = \pm E/c$.
photon #2: energy = E_0, momentum = $-E_0/c$.
The minus sign here is because this one is traveling in the opposite direction.
The Box: energy = m_0c^2, momentum = 0.
To find the total mass m, we will use the 'modern definition of mass:'

$$m^2c^4 = E^2 - p^2c^2.$$

The point here is that energies add and momenta add, but masses do **not** add. So, the total energy is $m_0c^2 + 2E_0$, and the total momentum is zero. Thus, the total mass is

$$m = \sqrt{E^2/c^4 - p^2/c^2} = m_0 + 2E_0/c^2.$$

6-4. The numbers in this problem simplify if we note that the mass of a proton (m_p) is roughly 2000 times the mass of an electron (m_e). So, $m_p = 2000m_e$.

This means that, if we create a proton and an antiproton at rest, the total energy is $(m_pc^2 + m_pc^2) = 4000m_ec^2$. Now, the proton/anti-proton pair are to be made from the positron/electron pair. So, energy conservation tells us that the energy of the positron/electron pair must be equal to the energy of the proton/anti-proton pair.

Now, the electron and positron are moving with some speed v. So, each has an energy $E = \frac{m_e c^2}{\sqrt{1-v^2/c^2}}$. This means that the total energy is $E_{total} = \frac{2m_e c^2}{\sqrt{1-v^2/c^2}}$. As we have said, this must be equal to the energy in the proton/anti-proton system. So, we have

$$4000 m_e c^2 = \frac{2m_e c^2}{\sqrt{1-v^2/c^2}}.$$

The factors of $2m_e c^2$ cancel nicely and we have:

$$(2000)^{-2} = 1 - v^2/c^2.$$

In other words, $v = c\sqrt{1-(2000)^{-2}} = (.999999875)c$.

To find the associated boost parameter, we take $\theta = \tanh^{-1}(v/c) = \tanh^{-1}(.99999975) \approx 8$.

6-5. [optional] a) At rest, $m_0 = E/c^2 = (1.6 \times 10^{-19} J)/(3 \times 10^8 m/s)^2 = 1.8 \times 10^{-36} kg$.

b) The rest mass of an electron ($9.11 \times 10^{-31} kg$) is roughly 5.1×10^5 times the mass for one electron-Volt. So, the rest mass of the electron is equivalent to 5.1×10^5 eV, or 510,000 eV.

c) When the electron accelerates through 5000 Volts, it gains 5000 eV of energy. So, its total energy becomes 515,000 eV. Thus,

$$E = \frac{m_0 c^2}{\sqrt{1-v^2/c^2}} = 515,000 eV.$$

Since $m_0 c^2 = 510,000 eV$, we have $\sqrt{1-v^2/c^2} = 510/515$. Thus, $1 - v^2/c^2 = (510/515)^2 = .98$, and $v^2/c^2 = (1 - .98) = .02$. Thus, $v/c = \sqrt{.02} = .14$ and $v = .14c$.

d) If the electron is moving at $.14c$, it's momentum is

$$p = \frac{m_0 v}{\sqrt{1-v^2/c^2}} = (9.11 \times 10^{-31} kg)(.14)(3 \times 10^8 m/s)(1/.98) = 3.9 \times 10^{-23} kgm/s.$$

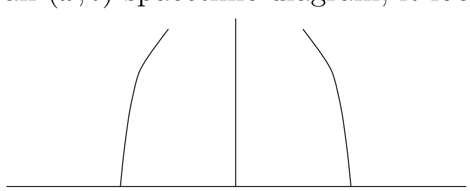
Using the Newtonian formula, we would have

$$p = m_0 v = (9.11 \times 10^{-31} kg)(.14)(3 \times 10^8 m/s) = 3.8 \times 10^{-23} kgm/s.$$

So, the relativistic effects in your TV set are big enough that the actual momentum of the electrons striking the screen is about 2.5% greater than what Isaac Newton would have expected. Now, it is important to know the momentum of the electrons when you are figuring out where they will hit the screen. If you were designing a monitor from first principles and didn't know about relativity, the electrons would hit the screen roughly 2.5% away from where you expected. On a screen which is one thousand pixels wide, this would mean that the electron would miss its target by 25 pixels. As a result, relativity is important in high resolution television and computer graphics! (Surprised???)

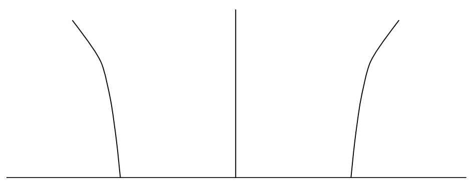
7-1. Although the goal of this problem is to look at things from the perspective of the (freely falling) middle stone, we are most familiar with looking at this situation from the reference frame of the earth. So, let's start thinking about it from the earth's perspective and then *translate* what we figure out into the reference frame of the stone.

A) So, what happens if we drop the three stones?? We know that each of them will fall toward the center of the Earth. The question is, what does this mean for their separation in the x direction? Well, the stones start off at rest relative to each other, but then come closer together. In other words, the stones start off with no relative velocity, but their relative acceleration of the stones (along the x direction) is *towards* each other. If we sketch this on an (x, t) spacetime diagram, it looks like:



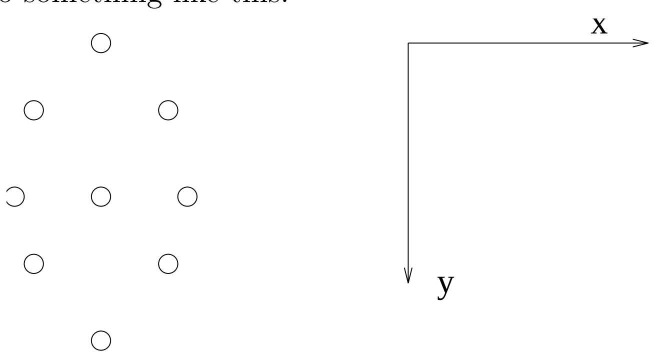
Now, would it make any difference if the stones started off with an initial velocity? Not if they all had *the same* initial velocity. Then their *relative* velocity still starts off at zero and increases as shown above.

B) Once again, let's start by thinking things through from the earth's perspective. In the language of Isaac Newton, the gravitational acceleration is greatest near the earth, so the lower stones will accelerate down at a greater rate than the upper stones. This means that, while the relative velocity of the stones starts at zero, their relative acceleration (in the y direction) is *away* from each other and the stones get farther apart as time passes. If we sketch this on an (y, t) spacetime diagram, it looks like:



Again, an initial velocity doesn't change anything, as long as all of the stones are given *the same* initial velocity.

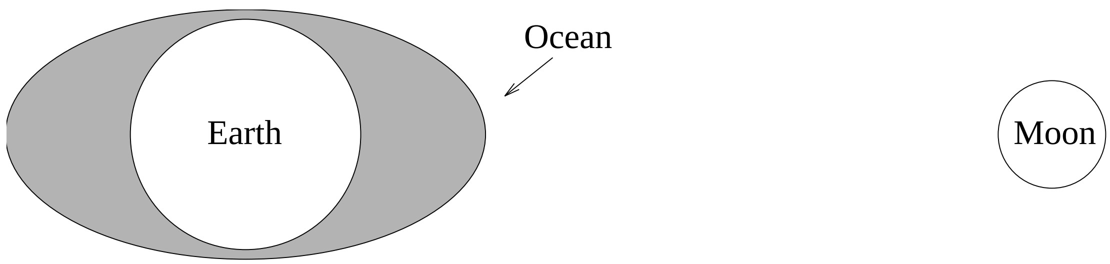
C) Putting the above two parts together, the stones on the sides will accelerate toward the central stone, while the stones on the top and bottom will accelerate away from the central stone. The configuration becomes distorted into something like this:



and the amount of the distortion increases as time passes. Now, are the stones close enough together that we can describe them by 'local' measurement? Recall what we meant by 'local.' We said that a region of spacetime is small enough to be considered as 'local' when the freely falling objects act *just* like inertial objects in gravity-free spacetime. Now, all of our stones above are freely-falling. But, as we have just said, some of them accelerate toward each other, while some of them accelerate away from each other. This is *not* like inertial objects in gravity-free spacetime, as such inertial objects always have *zero* relative acceleration! So, this system is too big to be considered 'local.'

If it were much smaller, and the stones were much closer together, then the relative accelerations would be too small to notice. In that case, our system would be small enough to count as 'local.'

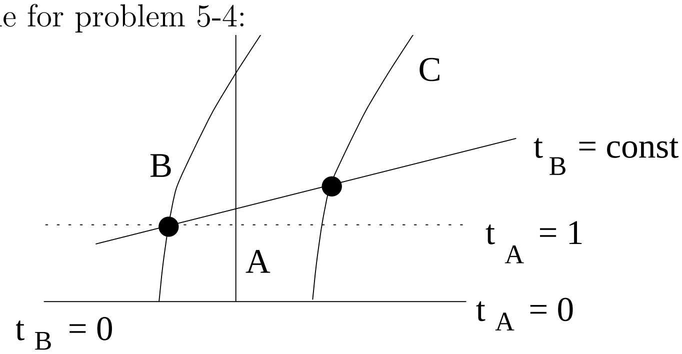
7-2. The ocean around the earth will be distorted just like the outer circle of stones in 1.C. around the central stone. Of course, in this case there are other phenomena (like the gravitational field of the earth itself) that keep the distortion from getting too big. Still, the shape is the same. The ocean around the earth looks like:



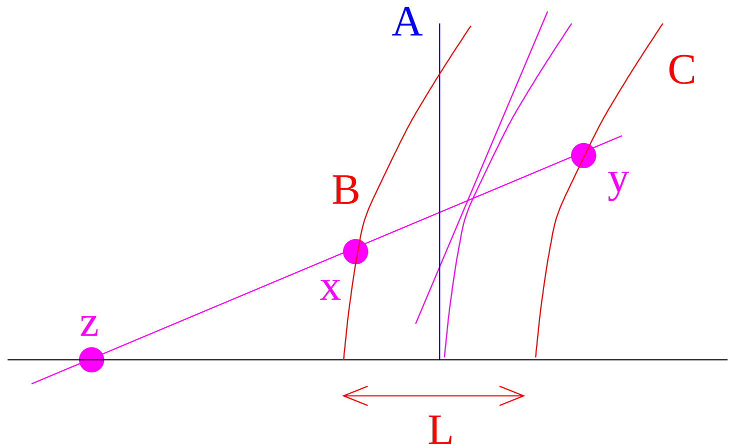
The thicker parts of the ocean manifest themselves as high tides as the earth turns under the moon, while the thinner parts of the ocean manifest themselves as low tides. The above picture makes it clear why there are two high tides and two low tides every day as the earth rotates within its envelope of water.

Solutions to Extra Optional Problems on Special Relativity

5-7. Let us start this problem, as with all problems, by drawing a spacetime diagram showing us all of the important features. It is best to draw such a diagram in an inertial frame, so let us use the frame of rocket A (the one that does not accelerate). The diagram, of course, looks much like the one for problem 5-4:

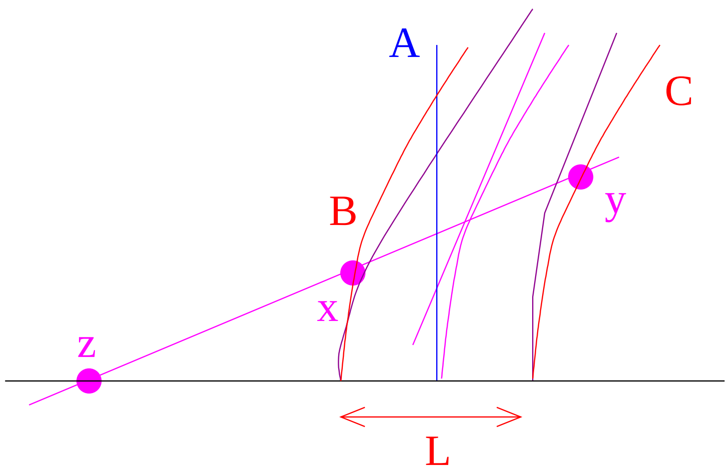


Now, we have a good intuition for strings moving at much less than the speed of light, but the statement of this problem indicates that things may be more complicated when they are moving at near the speed of light. So, as with many problems, what we would like to do here is to change reference frames in such a way that we can work with a slowly moving string. That is, let us consider things from the instantaneous rest frame of an atom, say, in the middle of the string. Rather than re-draw the diagram using that frame of reference, let me simply add three lines to the diagram above: the (curved) worldline of the central atom, a straight line that is instantaneously co-moving with that atom at some event, and the corresponding line of simultaneity. One might think that the 'kindest' thing to do to the string is to make sure that each atom on the string has the same acceleration as measured in A's reference frame (otherwise, A will find some part of the string to be stretched in this process). This would mean that the middle atom follows a worldline just like those of rockets B and C, but located half-way in between them.



So, what do things look line from the atom's reference frame? In particular, how far apart are the rockets in that frame? This is just the question: "What is the proper distance between the two events (x,y) marked on the diagram above?" In particular, is it larger or smaller than the original length (L) of the string?

To answer this question, let us draw two curves that begin at the same places as rockets B and C, but which actually do remain a constant proper distance L/2 on each side of our atom. It is not hard to draw such curves, as they are just the hyperbolae that remain a constant proper distance from the event marked z on the diagram. In particular, the one that begins with rocket B is more curved and the one that begins with rocket C is less curved:

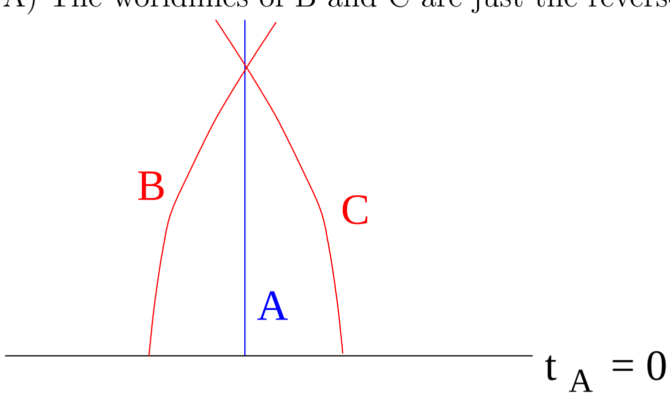


We can see that the proper distance between these new curves (which, by construction, always remains equal to L) is less than the proper distance between events x and y. As a result, in a frame of reference in which the atom is at rest (where we should understand how the atom will react!!!), the string has been stretched!

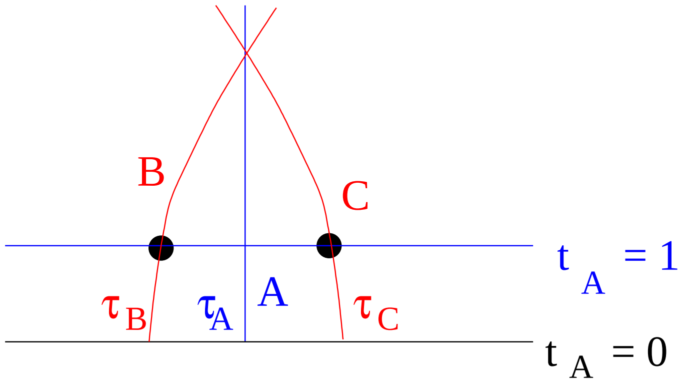
One can quickly check that this is true not just for the central atom, but in fact for any atom whose acceleration matches that of rockets B and C. As a result, the string as a whole feels that it has been stretched. When a weak string is stretched, it breaks. So, this must be what happens here. In other words "yes, the string really does break."

5-8. This problem is much like problem 5-4, but with one rocket accelerating in the other direction. We will attack it in a similar way.

A) The worldlines of B and C are just the reverse of one another:

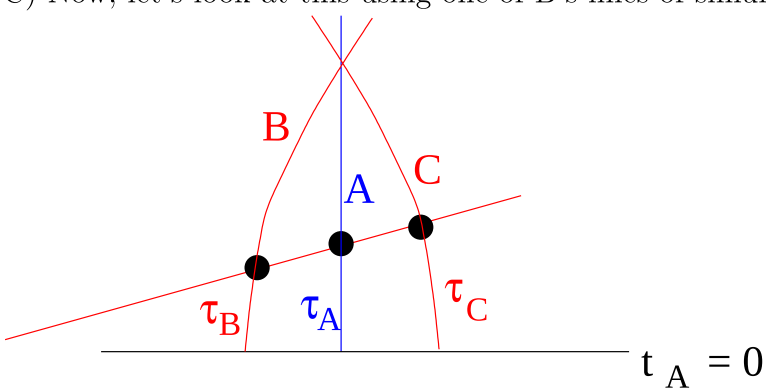


B) I've drawn one of A's lines of simultaneity below and marked off the relevant segments of the three worldlines. What we need to do is to compare the three proper times τ_A, τ_B, τ_C associated with these three segments.



As you can see, the part of B's worldlines between $t_A = 0$ and $t_A = 1$ is identical to the corresponding part of C's worldline, except that they are reversed. Thus, they correspond to the same proper time. During this time interval, both worldlines B and C always correspond to a higher speed than that of worldline A as measured in the inertial reference frame shown. As a result, less time elapses along worldlines B and C. Thus, we have $\tau_A > \tau_B = \tau_C$

C) Now, let's look at this using one of B's lines of simultaneity:



Here we have drawn the diagram at a fairly "early" time, before the rockets get moving too fast. Laster we will discuss when happens at a "late" time.

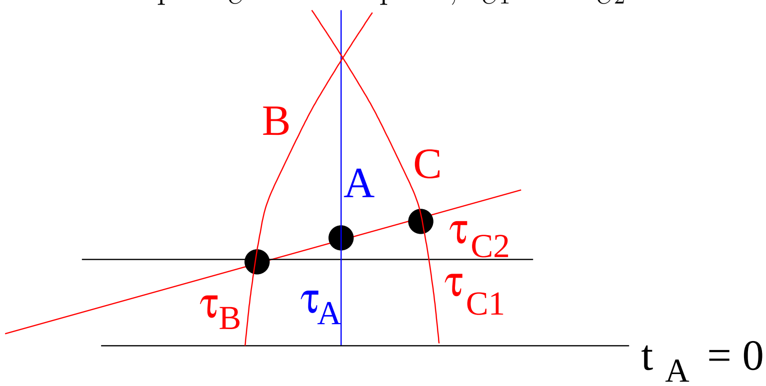
Let us first compare τ_A and τ_B. Recall that the proper time along any segment of a worldline is given by

$$\tau = \int_{t_0}^{t_1} \sqrt{1 - v^2/c^2} dt.$$

Note that the relevant part of A's worldline is taller than that of B's, which means that τ_A is associated with an integral over a larger t-interval than is

τ_B. Also, the velocity is smaller along A's worldline, so that $\sqrt{1 - v^2/c^2}$ is greater along A's worldline. Thus, we see that $\tau_A > \tau_B$.

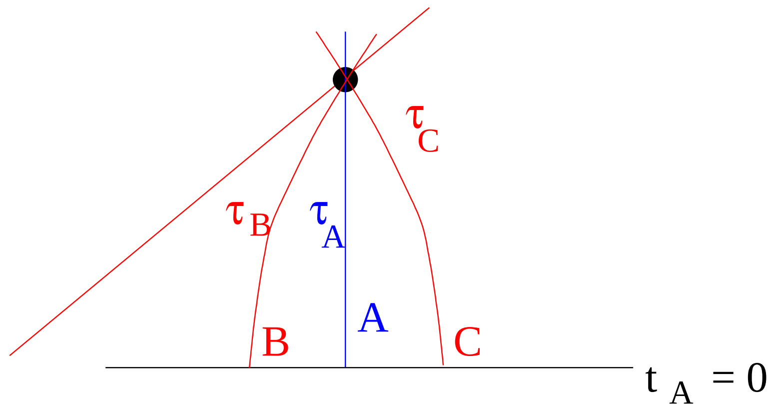
Now, how about τ_C? Well, to sort this out, let us draw in another reference line and split τ_C into two parts, τ_{C1} and τ_{C2}:



The point to notice here is that the part of C's worldline corresponding to τ_{C1} is just the reverse of the part of B's worldline corresponding to τ_B. Thus, we must have $\tau_B = \tau_{C1}$! Thus, $\tau_C = \tau_{C1} + \tau_{C2} > \tau_B$.

Hmmmm.... but is τ_C greater than τ_A? After all, there are two competing effects: C is moving faster than A, but we are interested in a 'taller' part of C's worldline. For small velocities the answer is yes (i.e., $\tau_C > \tau_A$), but it takes more work to see this. Basically, one needs to study the integral $\tau_C = \int_{t=0}^{T} \sqrt{1 - v^2/c^2}$ for small t_B. For small t_B, the velocities v_B and v_C are also small. The square root factor ($\sqrt{1 - v_C^2/c^2}$) encodes the effects of the speed. Expanding it in a Taylor's series, one sees that the effect is quadratic in the velocity of C (and is therefore quadratic in the velocity of B since, for small v_C, we have $v_B \approx v_C$). On the other hand, calculating the value of T encodes the effect of looking at a taller part of C's worldline. Since this follows just from the slope of B's line of simultaneity, it depends *linearly* on v_B. For small v, a linear effect is always larger than a quadratic one. So, the effect of being taller wins out over the effect of being faster and, for small times (so that v_B is small) we have $\tau_C > \tau_A$.

However, as in the famous problem 5-4, things change if we look at a late time. Consider for example B's line of simultaneity through the event where the three rockets meet:



Here, it is clear that $\tau_B = \tau_C > \tau_A$, since all three worldlines are the same height ("tallness") but A has zero velocity whereas the others have a nonzero velocity.

If we wait even longer the worldlines of A and C will get cutoff by B's acceleration horizon.... B will see their clocks 'freeze' at the value where A and C cross this horizon, while B's own clock will continue to tick. As a result, for a very late time we have $\tau_B > \tau_A > \tau_C$.

D) Since there is a symmetry relating B and C, C's measurements are in direct parallel with B's just discussed in part (C), but with "rocket B" everywhere replaced by "rocket C" and vice versa.

E) Rocket B finds A to move closer, and then they all meet up at the event marked in the last diagram in part (C). Then rocket A passes, and approaches B's acceleration horizon. At this point, it 'freezes' at B's acceleration horizon and asymptotically approaches this horizon. The horizon, of course, is at a proper distance of c^2/α_B from rocket B. Thus, rocket B will find rocket A to be a distance c^2/α on the left after a very long time. Note that this would be true of both the front and back of A, with the result that rocket B finds A to shrink to zero size as time passes. Much the same is true if B watches rocket C, or if C watches A or B.

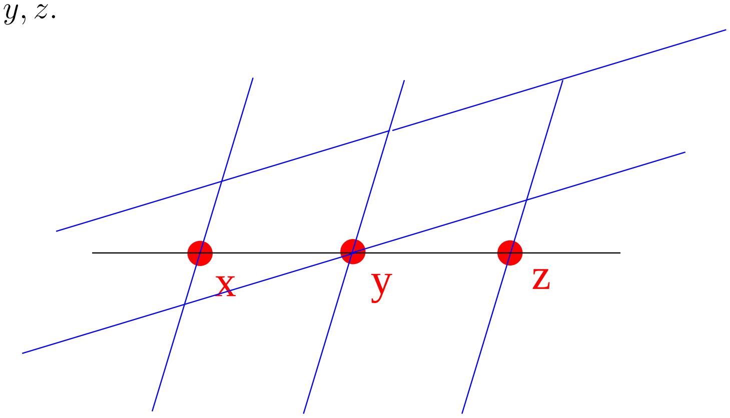
5-9. A) The radius r does not change but, by symmetry, the wheel must remain a wheel in the original inertial frame. Thus, the edge is a circle of radius r, and the circumference is $C = 2\pi r$.

B) Let us first realize that we could measure the distance in part A by holding a number of stationary meter sticks above the wheel. Let us now consider a meter stick moving at $.8c$. If we compare this to one of our stationary meter sticks, it will be shorter. Thus, we can put more than one meter stick in between the two ends of our stationary stick. As a result, over a distance which is $2\pi r$ in the original inertial frame, we will be able to tack down $\frac{2\pi r}{\sqrt{1-v^2}} = 10\pi r/3$ meter sticks.

C) Well, suppose that the ends of adjacent spokes are 1m apart when the wheel is at rest. After the wheel is up to speed, we saw in part B that, in the co-moving frame, they are now $\frac{5}{3}m$ apart! When a thread that is $1m$ long is stretched to $\frac{5}{3}m$, it breaks.

D) As viewed from the inertial frame, each clock is the same as any other – just at a different point around the circle. So, from this perspective the same thing must happen to each clock. That is, they must all run identically. Thus, they remain synchronized *with each other* as viewed from this frame. However, due to the usual time dilation they must begin to run slower than a clock that remains inertial.

E) Well, your notion of simultaneity does *not* correspond to that of the inertial frame. To see what this means, let us draw a spacetime diagram using the inertial reference frame showing your clock, and the ones immediately ahead of and behind you. If the circle is large and there are enough clocks, these clocks are all effectively in a straight line along the direction of the velocity, so we can draw these on a 1+1 dimensional diagram. (Note, however, that the inertial clock will not appear as it is located somewhere off the page from the perspective of this diagram.) The important thing to remember is what we found in part D: in the frame of reference of our diagram below, the three clocks remain synchronized. Thus, they read the same values at events x, y, z.



Two lines of simultaneity for our clock are also shown. It is clear that, along the lower line of simultaneity, the clocks read different values! As a result, they are not synchronized.

Since they began synchronized and now are not, at some time the clocks must have run at different rates. However, let us study the three worldline segments determined by the two lines of simultaneity. The three segments are identical! Thus, they each correspond to the same proper time, and each clock finds its neighbors to run at the same rate as it does. So, the clocks appear to each other to run differently during the process when the wheel is getting up to speed, but then appear to run at the same rate.

5-10. Note that this problem is basically the same as number 5-9, but just inverted.

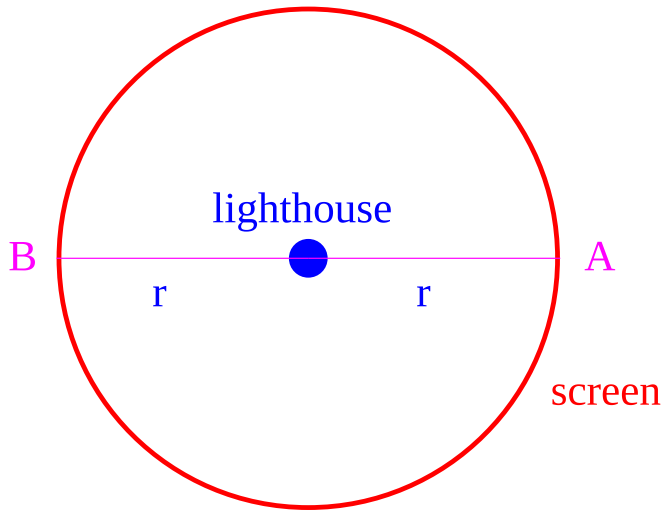
A) This is like problem 5-9B. The workers measure a circumference $C = \frac{2\pi r}{\sqrt{1-v^2/c^2}} = \frac{10\pi r}{3}$.

B) Note that the workers farther out are moving faster and therefore experiencing more time dilation. So, at the end of the 'day' when they all get back together, the ones who worked farther out have aged less.

C) From problem A we see that we built a circle out of $\frac{10\pi r}{3}$ worth of boards at a distance of radius r. When we stop the merry-go-round, we are forcing this onto a circle of radius r and circumference $2\pi r$. This clearly will not work – there are too many boards! So, the merry-go-round will have to buckle to fit the extra boards into too small of a space.

5-11. An observer far away (at a distance r) indeed finds the spot to sweep past at a speed $v = \frac{2\pi r}{T}$ where T is the time it takes the lighthouse to turn around once. For large enough r, this is greater than c.

However, this is not a problem for special relativity. The point is that nothing is really moving with this spot on the screen. Instead, the light is moving *outward* from the center at exactly c. One typically says that "no *information* can travel faster than the speed of light." Here, the spot does not convey any information between two points that it passes. Suppose, for example, that you are at one point (A) on the screen and you wish to influence the spot as it appears at another point (B) on the screen as shown below.



If r is very large then, after the spot arrives where you are, there is nothing that you can do to influence how the spot appears at B! The point here is that, if r is very large, then when the spot arrives at A the photons that will make the spot at B are already on their way from the lighthouse to B! Thus, the only way to influence the spot at B would be to catch these photons. However, you are at A and the photons you need to catch are moving away from you at the speed of light. So, no luck.