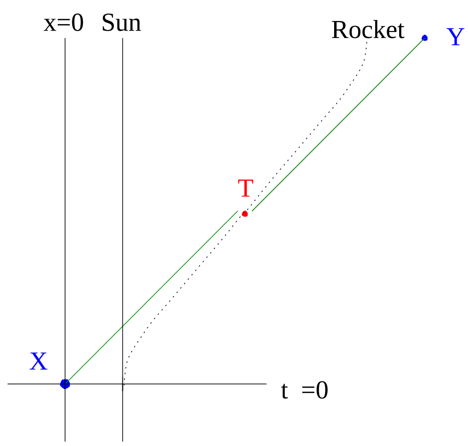
Solutions to Homework Assignment #6 – PHY312

5-1. a) This first part is just a question of converting units. Let us first note that one light year is $c(1year) = (3 \times 10^8 m/s)(\pi \times 10^7 sec) \approx 10^{16} m$. The fact that $1year$ is very close to $\pi \times 10^7$ seconds is a very useful one to remember and is known to most physicists. Of course, you can also calculate the number of seconds in a year by multiplying the number of seconds in a minute, times the number of minutes in an hour,... etc. The answer is very close to $\pi \times 10^7$.

Anyway, this tell us how to convert meters to light years. So, we may write

$$g = 10m/s^2 = 10\frac{m}{s^2}\left(\frac{1lightyear}{10^{16}m}\right)\left(\frac{1year}{\pi \times 10^7 sec}\right)^2 \approx 1\frac{lightyear}{year^2}.$$

b) OK, the two halves of the trip (speeding up and slowing down) are basically the same, except that 1) the accelerations are in opposite directions and 2) the first half *starts* at rest while the second half *ends* at rest. So, the spacetime diagram should look like this:




Here, the dotted line is your worldline – the worldline of the rocket. I have marked the event T where your rocket turns around and begins to decelerate. During the first half of the trip (before T) and the second half (after T), the rocket's worldline is a piece of some hyperbola. The two pieces have exactly the same shape, except that the second one is 'upside down and backwards' relative to the first. During the first half of the trip, the rocket maintains a constant proper distance (1 light-year) from event X, but in the second half of the trip, it maintains a constant proper distance from a different event (Y). The solid lines coming out of X and going into Y are the light rays which would have formed the asymptotes of the two hyperbolae had the acceleration or deceleration continued forever.

c) Note that there is a symmetry that connects the two halves of the trip. So, the distance traveled (as measured by the Sun) in the two halves of the trip must be the same. In the first half, your acceleration is uniform, so we can use those nice equations (say, 5.17 or 5.18 together with 5.6) from class: we start 1 light-year from event X (which lies at $x = 0$), and at proper time $\tau$ our distance from $x = 0$ is $x = (c^2/\alpha)\cosh(\alpha\tau/c)$. At event T, $\tau = 10 years$. Since $\alpha = 1g = 1(light - year)/year^2$, we have $x = (1light - year)\cosh(10) = 11,013(light - years)$. So, at event T, you have actually *traveled* $11,013 - 1 = 11,012$ light-years. Therefore, during the entire trip, you travel $22,024$ light-years. This is most of the distance toward the center of the galaxy. For comparison, the center of the galaxy is about $30,000$ light years away. A quick calculation shows that one could therefore reach the center of the galaxy by accelerating at $1g$ for 10.3 years of proper time, turning around, and decelerating for another 10.3 years.

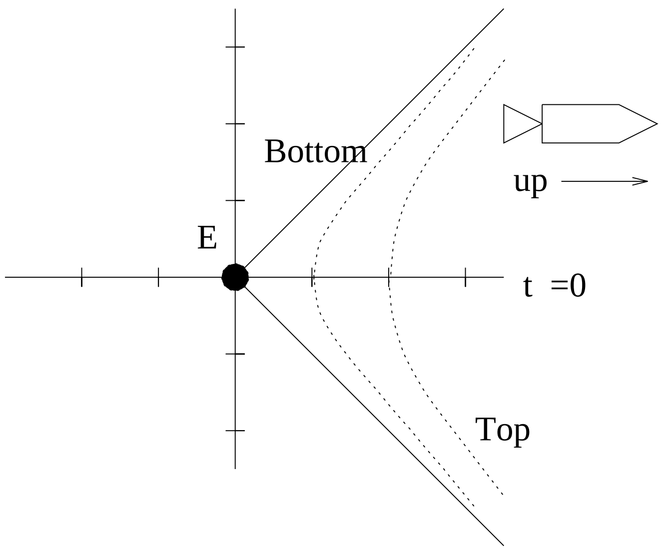
d) As measured by the sun, the time of event T is $(1year)\sinh(10) = 11,013years$. So, when you finally stop, folks back home think that you have been traveling for twice this: $22,026$ years.

e) This is just asking for the boost parameter ($\theta$) and velocity ($v$) at event T. Recall that $\theta = \alpha\tau/c = 10$, while $v = c\tanh\theta = c\tanh(10) = (.9999999959)c$.

5-2. OK, so we're at the top of a rocket which is half a light year tall and our proper acceleration is $1g = 1(light - year)/year^2$.

A) Since we accelerate uniformly, at $1(light - year)/year^2$, we remain a constant proper distance $c^2/g = 1(light - year)$ from some special event – let's call this event E. And, it is important to remember that event E is *below* us (if we refer to the direction in which we accelerate as 'up').

The bottom of the rocket is 1/2 a light year below us. So, it remains a constant proper distance of $l = 1lightyear - (1/2)lightyear = (1/2)(lightyear)$ above event E. This is shown in the following diagram:




The top and bottom of the rocket are the two dotted lines, while the solid lines coming out from (and going into) event E are the light rays which form the asymptotes of the hyperbolae followed by the top and bottom of the rocket. The coordinate $t$ refers to some (unspecified) inertial observer. Each of the tick marks along the $t = 0$ line represents half of a light year.

The proper acceleration at the bottom of the rocket is therefore $\alpha_{bottom} = c^2/l_{bottom}$, where $l_{bottom}$ is the (constant!) proper distance between the bottom of the rocket and event E. We have:

$$\alpha_{bottom} = c^2/l_{bottom} = \frac{1(light - year)^2/year^2}{(1/2)(light - year)} = 2(light{-}years)/year^2 = 2g.$$

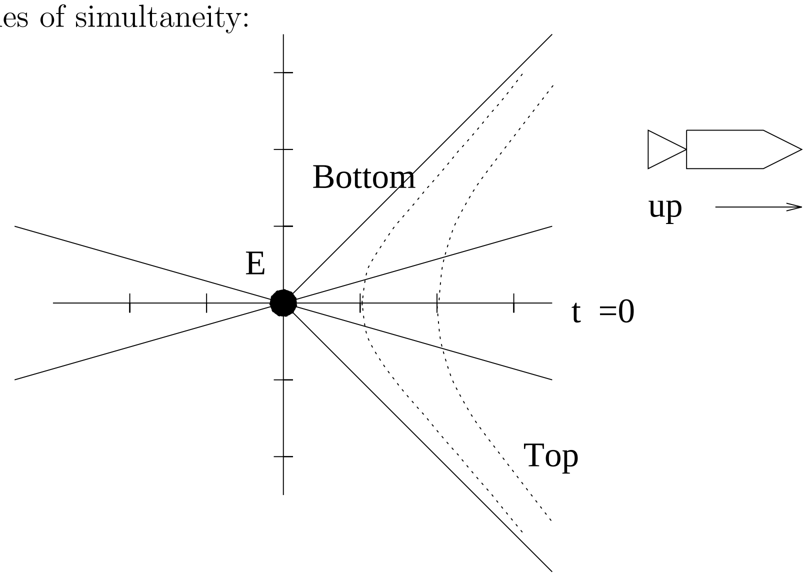
B) Recall that, for two objects (1 and 2) that both follow curves of constant proper distance from the same event E (and which are on the same side of E), the rates at which their clocks run are related to their proper accelerations by

$$\alpha_1 \tau_1 = \theta/c^2 = \alpha_2 \tau_,$$

since we found that both top and bottom have the same boost parameter (i.e., the same velocity) along a given line of simultaneity.

From our diagram, the top and bottom of our rocket constitute two objects that move along such curves, and the bottom of the rocket has a proper acceleration which is twice as great as that of the top. Therefore, the bottom clock must tick half as fast as the top one. In other words, if one second passes at the bottom of the rocket, two seconds will pass at the top.

c) For this part, we just need to redraw the diagram above with a few lines of simultaneity:




The lines of simultaneity are the new solid lines above – they pass right through event E. The important thing here is that these are the lines of simultaneity for both the top of the rocket and the bottom of the rocket. To actually label these lines with some time $\tau$ (say, as measured by us) would require a little calculation. (This wasn't in the original problem, but I'll do it for completeness). One of the lines of simultaneity passes through the worldline of the top of the rocket at about $t = (5/8)(years)$. Now, for uniform acceleration, $t = (c/\alpha)\sinh(\alpha\tau/c)$. Or, solving this for $\tau$, $\tau = (c/\alpha)\sinh^{-1}(t\alpha/c)$. So, $\tau = (1year)\sinh^{-1}[(5/8)] = .59years$ for the top of the rocket. (Of course, the clock at the bottom ticks only half as fast. So, according to the bottom clock this same line is $\tau_{bottom} = .295years$).

d) These last two parts are just to make a point about what happens if part of the rocket gets close to the acceleration horizon. If the rocket is 3/4 of a light year tall, then the bottom is only 1/4 of a light year from the special event E. We have

$$\frac{\alpha_{top}}{\alpha_{bottom}} = \frac{l_{bottom}}{l_{top}},$$

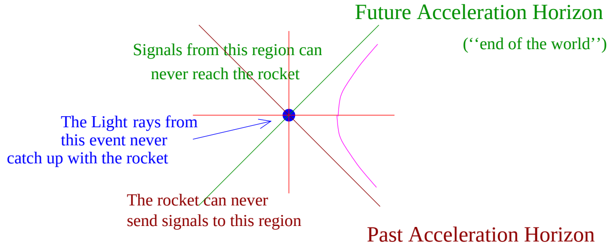
where $\alpha$ is the proper acceleration and $l$ denotes the proper distance to event

E.

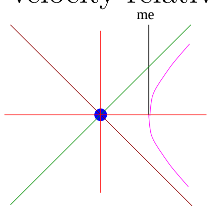
Since the bottom is four times closer to E than we are, it has four times the acceleration: $4g$.

e) Now the bottom is only (1/100)th as far from event E as we are. So, the acceleration is $100g$.

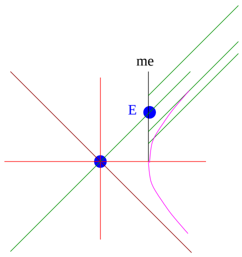
5-3. a) This is just the diagram of from the front of section 5.2.1 in my notes:




b) Now, we need to add our worldline. After we are kicket out of the rocket ship, what kind of worldline do we follow? Well, what forces act on us? The answer is none – we are inertial observers. Thus, our worldline is straight. We just keep moving with whatever velocity we had when we were tossed out. In the diagram below, I have chosen to have us tossed out at $t = 0$, and we have been tossed out softly so that, when we leave the rocket, our velocity relative to the rocket is zero.

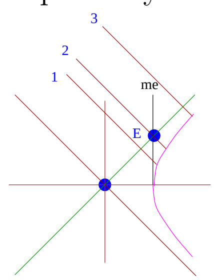



c) This is like the Alphonse and Gaston problem from chapter 4. To determine what they see as they watch you (that is, as they receive light rays from you), we should draw in a few of these light rays. This has been done below:



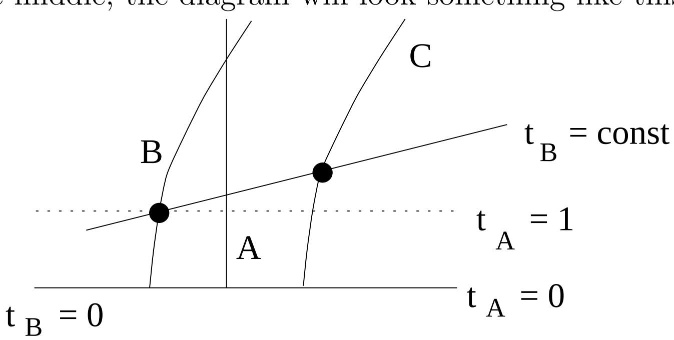

Note that some of the light rays you emit (those from after you cross the horizon) *never* reach the rocketship!!! As a result, no matter how long they watch you they never see you age past the event $E$ (marked above) where you cross the horizon. So, they see a finite amount of your life stretched out over their eternity.... In other words, they see you *extremely* slowed down.

d) Now we just draw a few light rays coming from them toward you. Note that no matter when they emit the light ray, it does in fact eventually catch up to you. So, you do see their entire life (in contrast to them seeing only a finite part of yours).

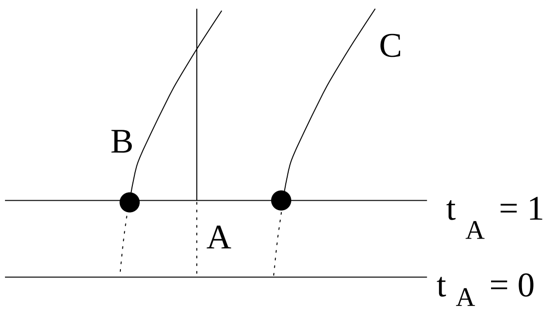



However, you still see them slowed down. Since they are moving very fast, there is a lot of time dilation and, if they wait one year of their *proper* time to emit a light ray, this could mean that in your reference frame they wait 10,000 years or more (as in problem 5-1)!!! Even after they emit the light ray, it still has to travel toward you (from, perhaps, 10,000 light years away...). So, while you do eventually receive the light ray, it takes a very long time to arrive.

5-4. a) Drawing the spacetime diagram holds the key to the entire problem. Recall that the reference frame of Rocket A is inertial, and that the worldline of B will look very much the same (in this reference frame) as that of C. The only difference will be that they are separated by some distance which, in this frame, will remain constant. Since Rocket A starts exactly in the middle, the diagram will look something like this:



b) Now, to answer the question about how the times compare, we must draw in some line of simultaneity. Since this part is from A's perspective, we will need some of A's lines of simultaneity. In particular, if we want to figure out how fast A finds the three clocks to run, say, between $t_A = 0$ and $t_A = 1$, we need to draw these two lines on the diagram. This looks like:

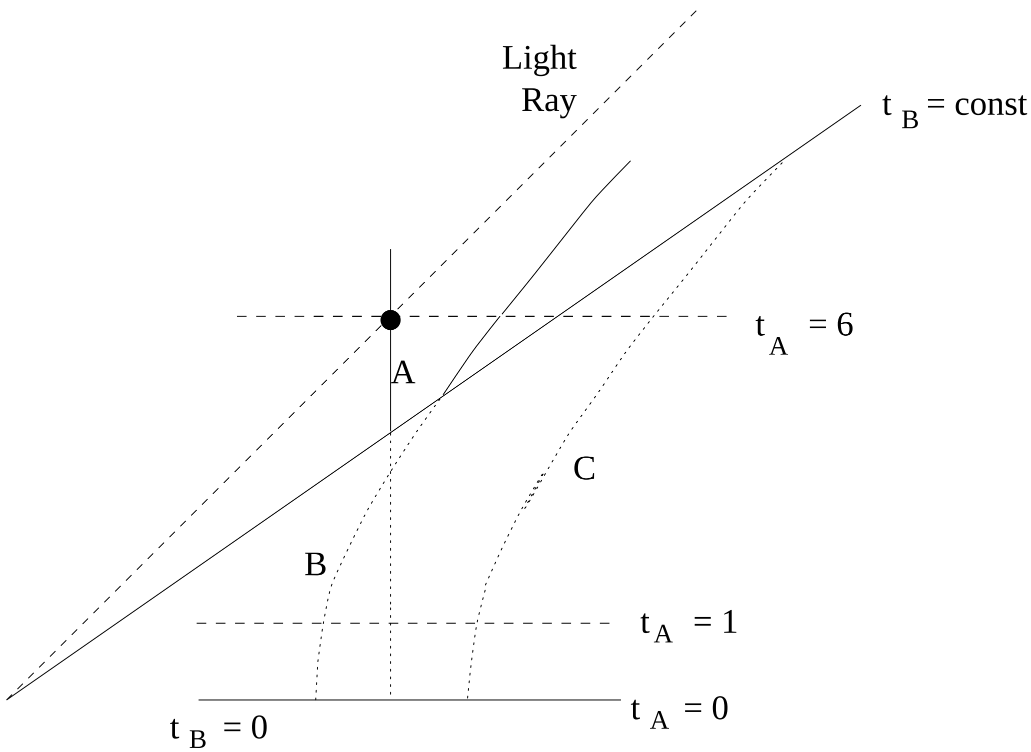



The question of how fast A finds the clocks to run is just the question of how much time the three clocks measure to pass along the dotted segments of the worldlines shown above. That is, the clock which runs the fastest as determined by A is the one that measures the most time to pass between the two lines of simultaneity. This is just the question "Which of the dotted segments above represents the greatest proper time, and which is the least?"

Note that the pieces of the worldlines of B and C between $t_A = 0$ and $t_A = 1$ are *identical*. So, they must represent *the same* proper time. Thus, as determined by A, clocks B and C run at the same rate.

How does this rate compare to A's own clock? Well, A is just an inertial observer watching some moving clocks, so time dilation tells us that A must find the other clocks (B and C) to run more slowly. We can see this from the graph because, while the three dotted segments all reach the same height on our diagram, the segment for A is straighter and more vertical (i.e., moving more slowly). Recall that straight vertical lines have the greatest proper time.

c) Let us follow the hint and suppose that B waited a long time before making his measurements. Then the line of simultaneity for B would be high up the diagram and the slope would be close to 45 degrees (the speed of light):

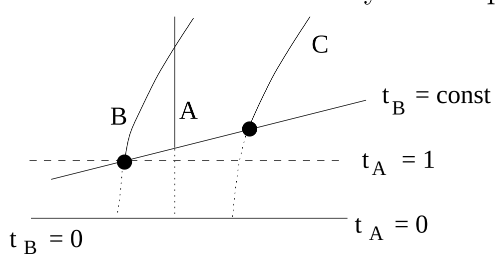



Now, the readings on the clocks at $t_B = const$ are determined by the amount of proper time that passes along the various dotted parts of the worldlines. (Recall that a clock does indeed measure proper time along the worldline it travels.) Note that the dotted piece of B's worldline is *taller* than that of A. But of course, B is moving, so time dilation means that B's worldline represents less proper time than we might usually think. So, how do we decide which actually has the greatest proper time?

If we wait until a very late time as measured by B, then the time that B's own clock read will be very large (nearly infinite). However, at very late times, B's lines of simultaneity approach the light ray that forms the asymptote of B's hyperbolic worldline. This means that, no matter how long B waits, the corresponding line of simultaneity will never go any farther up A's worldline than does the light ray shown as a dotted line above. It always crosses A's worldline a little bit below the event marked with the big dot. That dot is located at $t_A = 6$. So, no matter how long B waits, rocket B always finds that A's clock reads less than 6! Since B's clock reads some very large value, B finds A's clock to be running much more slowly.

And what does B think about C's clock? Consider the dotted parts of B's and C's worldlines shown above. The worldlines of B and C are identical, but more of C's worldline appears dotted (because more of it is below B's line of simultaneity). So, the proper time along the dotted part of C's wordline is bigger and B finds C's clock to run faster. To summarize: at very late times, B finds A's clock to run slower than B's, which runs slower than C's.

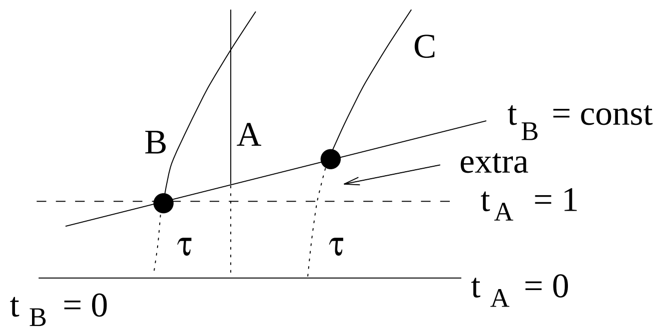
Now, we could also ask what happens at early times, close to $t = 0$. I did not ask you to work this out on the homework but, for completeness, I will do it here. Note that, at $t_A = 0$, B has not yet begun to accelerate and so has no velocity relative to A. As a result, the line $t_A = 0$ is also a line of simultaneity for B. On the other hand, (after $t = 0$) B is moving to the right, so that B's line of simultaneity will slope up and to the right like this:




Here, I am not sure exactly which line of simultaneity I have drawn (in the sense that I do not know what B's clock reads on this line), but this is not important. The question of whose clock B finds to run fastest in this time interval is again the question of which of the dotted line segments represents the most proper time.

Let us first compare B and A. The dotted segment of A's worldline both reaches higher on the diagram and is straighter, so it must be the longer proper time. Thus, at early times B finds A's clock to run faster than her own. Recall from part (b) that this agrees with A's conclusion.

Let us now compare B and C. We can make this easier by making use of the line $t_A = 1$ which I have drawn as a dashed line of the diagram above. The dotted segment of B's worldline is again identical to the part of C's worldline below $t_A = 1$ and both have the same proper time $\tau$. But, there is also an extra bit of C's worldline that contributes:

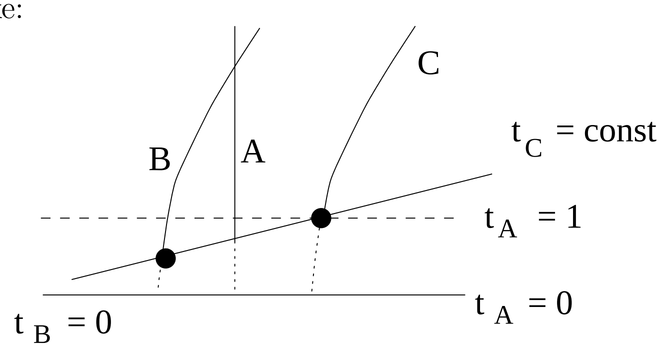



So, the dotted segment of C's worldline represents more proper time than does the dotted segment of B's worldline. Therefore, B finds C's clock to run faster.

Finally, we need to compare A's clock and C's clock from B's perspective. This is actually more subtle. Looking at the diagram, we see that the dotted part of C's worldline reaches a greater height, but that it is more curved than that of A and so experiences more time dilation. Which one of these effects is more important?

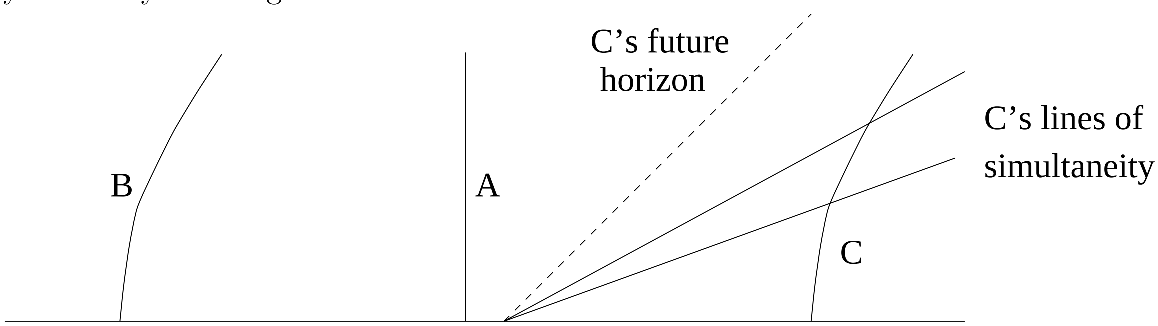
Close to $t_B = 0$ we can again read an answer off of the diagram. If we study the rockets at very early times, C has had very little chance to accelerate. So, it is moving very slowly and the time dilation factor is not very important. The fact that the dotted part of C's worldline reaches a greater height is more important (note that this factor becomes very big if C is far away!!!), and B finds C's clock to run faster than A's.

d) Now we ask the same question for C. A line of simultaneity for C looks like:



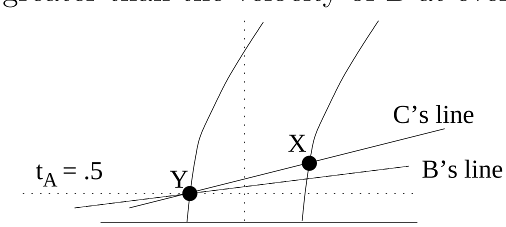

It is clear from the diagram that the same argument goes through and that rocket C agrees with rocket B in saying that C's clock runs faster than B's. For large times (which I asked you to consider) C will find A's clock to run even slower than B's while for very small times (which I did not ask you to worry about) C's clock will find A's clock to be between B's and C's. At least, this is the answer when B and C start fairly close together.

However, if you choose to start the rockets too far apart, C's lines of simultaneity may only cross B's and A's worldline outside of the wedge where they are really meaningful:




The result is that, according to C's lines of simultaneity, A's clock and B's clock are actually running backwards. However, strictly speaking C really can't observe A's clock or B's clock as they lie beyond C's future acceleration horizon.

This next bit is more than the problem asked, but, for completeness, I'll do it anyway It turns out that a more detailed argument shows that rockets B and C do not agree completely. While they both find C's clock to run faster, they do so by different amounts. You can see this from the diagram, too. Consider one of B's lines of simultaneity through the event where B's worldline intersects $t_A = .5$. The resulting line of simultaneity is *less* steep than C's line of simultaneity through that point, because the line we drew for C corresponds to a higher velocity. That is, the velocity of C at event X is greater than the velocity of B at event Y (see below).




Note that more of C's worldline lies below the solid line (C's line) than below the dashed line (B's line). So, rocket C finds the difference in the rates at which the clocks run to be even greater than rocket B does.

5-5. In the end, these problems come down to just doing a little calculus. Let's give the first one a try.

We are given that $v/c = \sqrt{1 - \frac{t_0^2}{t^2}}$. So, the change in proper time from $t = t_0$ is

$$\Delta\tau = \int_{t_0}^{t} dt\, \sqrt{1 - v^2/c^2} = \int_{t_0}^{t} dt\, \sqrt{\frac{t_0^2}{t^2}}.$$

Since $\tau = 0$ at $t = t_0$, we can write this as

$$\tau = \int_{t_0}^{t} dt \frac{t_0}{t} = t_0 \ln(t/t_0).$$

Since $\tau$ increases only logarithmically, the effects of time dilation are very large for large $t$. This is to be expected since the velocity approaches the speed of light.

Now we want to compute the proper acceleration. We start with the boost parameter

$$\theta = \tanh^{-1}(v/c) = \tanh^{-1}\left(\sqrt{1 - \frac{t_0^2}{t^2}}\right).$$

From the chain rule (and using the hint), we have

$$\frac{d\theta}{dt} = \frac{1}{1 - v^2/c^2} \frac{d}{dt}(v/c).$$

But, $1 - v^2/c^2 = t_0^2/t^2$ and $\frac{d}{dt}(v/c) = \frac{1}{2}(1 - t_0^2/t^2)^{-1/2}(2t_0^2/t^3)$. So, we have

$$\frac{d\theta}{dt} = \frac{t^2}{t_0^2}(1 - t_0^2/t^2)^{-1/2}\frac{t_0^2}{t^3} = t^{-1}(1 - t_0^2/t^2)^{-1/2}.$$

However, what we want is $\frac{d\theta}{d\tau}$. So, we need to use the chain rule again: $\frac{d\theta}{d\tau} = \frac{d\theta}{dt}\frac{dt}{d\tau}$.

Recall that $d\tau = \sqrt{1 - v^2/c^2}dt$, so $\frac{dt}{d\tau} = (1 - v^2/c^2)^{-1/2} = t/t_0$. Thus, we have

$\alpha(t) = c\frac{d\theta}{d\tau} = ct_0^{-1}(1 - t_0^2/t^2)^{-1/2}$.

Note that the rocket starts (at $t = t_0$) with no acceleration since $\alpha(t_0) = 0$. However, as $t \to \infty$, the last factor goes to one and $\alpha(t) \to c/t_0$. For large $t$, the proper acceleration becomes constant. So, this worldline represents a rocket whose acceleration begins at zero but increases smoothly and, after awhile, has an approximately uniform acceleration.

5-6. This one is very similar. Here, $v/c = \sqrt{1 - \frac{t_0^4}{t^4}}$, so $1 - v^2/c^2 = t_0^4/t^4$. As a result,

$$\tau = \int_{t_0}^{t} dt \; \frac{t_0^2}{t^2} = t_0^2 \left( \frac{1}{t_0} - \frac{1}{t} \right).$$

Notice that as $t \to \infty$, $\tau$ goes to a finite value $\tau = t_0$! So, a rocket moving along this worldline reaches $t = \infty$ after only a finite proper time! One might wonder: "What would happen to you at $\tau = t_0$ if you were actually on such a rocket ship?"

Before embarking on a calculation of the proper acceleration, let me take a moment to collect some of the results that we derived above:

$$c^{-1}\alpha(t) = \frac{d\theta}{d\tau} = \frac{d\theta}{d(v/c)}\frac{d(v/c)}{dt}\frac{dt}{d\tau} = \frac{1}{(1 - v^2/c^2)^{3/2}}\frac{d(v/c)}{dt}.$$

Now, $\frac{1}{(1-v^2/c^2)^{3/2}} = t^6/t_0^6$ and

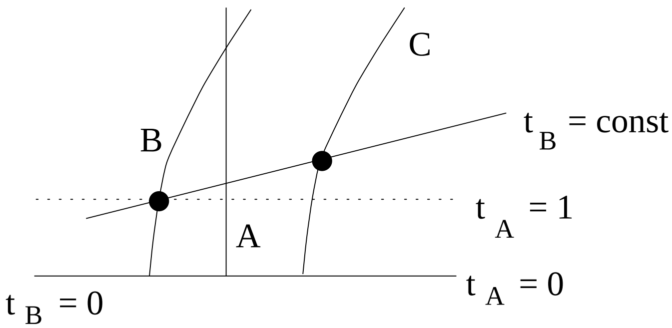
$$\frac{d(v/c)}{dt} = \frac{1}{2}(1 - t_0^4/t^4)^{-1/2}(4t_0^4/t^5).$$

So, we have

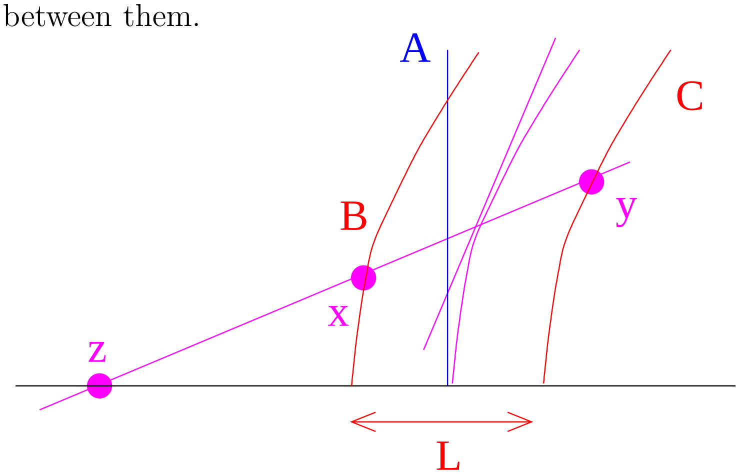
$$\alpha(t) = \frac{2t}{t_0^2}(1 - t_0^4/t^4)^{1/2}c.$$

Since the factor in parentheses goes to one for large $t$, the proper acceleration *diverges* as $t \to \infty$. In other words, this corresponds to a rocket whose acceleration becomes infinite in the far future. But this means that the force on any part of the rocket ship must also become infinite. So, it is physically impossible to build such a rocket ship: it would always tear itself (and any passengers) into pieces before it reached $t = \infty$ (or $\tau = t_0$).

Optional problem: 5-7. Let us start this problem, as with all problems, by drawing a spacetime diagram showing us all of the important features. It is best to draw such a diagram in an inertial frame, so let us use the frame of rocket A (the one that does not accelerate). The diagram, of course, looks much like the one for problem 5-4:



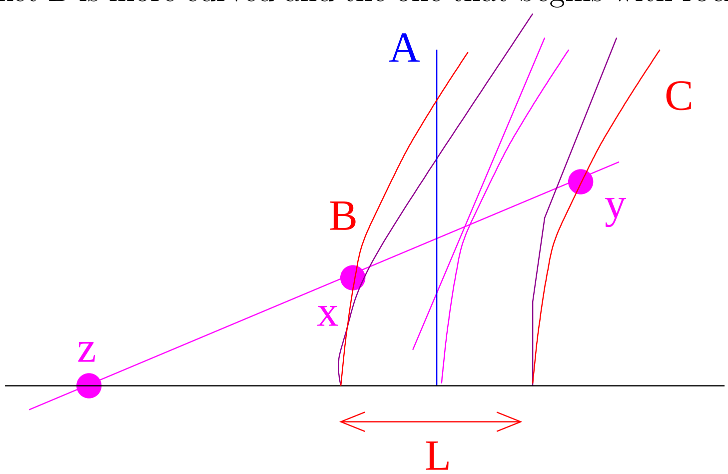

Now, we have a good intuition for strings moving at much less than the speed of light, but the statement of this problem indicates that things may be more complicated when they are moving at near the speed of light. So, as with many problems, what we would like to do here is to change reference frames in such a way that we can work with a slowly moving string. That is, let us consider things from the instantaneous rest frame of an atom, say, in the middle of the string. Rather than re-draw the diagram using that frame of reference, let me simply add three lines to the diagram above: the (curved) worldline of the central atom, a straight line that is instantaneously co-moving with that atom at some event, and the corresponding line of simultaneity. One might think that the 'kindest' thing to do to the string is to make sure that each atom on the string has the same acceleration as measured in A's reference frame (otherwise, A will find some part of the string to be stretched in this process). This would mean that the middle atom follows a worldline just like those of rockets B and C, but located half-way in between them.




So, what do things look line from the atom's reference frame? In particular, how far apart are the rockets in that frame? This is just the question:

"What is the proper distance between the two events (x,y) marked on the diagram above?" In particular, is it larger or smaller than the original length (L) of the string?

To answer this question, let us draw two curves that begin at the same places as rockets B and C, but which actually do remain a constant proper distance L/2 on each side of our atom. It is not hard to draw such curves, as they are just the hyperbolae that remain a constant proper distance from the event marked z on the diagram. In particular, the one that begins with rocket B is more curved and the one that begins with rocket C is less curved:




We can see that the proper distance between these new curves (which, by construction, always remains equal to L) is less than the proper distance between events x and y. As a result, in a frame of reference in which the atom is at rest (where we should understand how the atom will react!!!), the string has been stretched!

One can quickly check that this is true not just for the central atom, but in fact for any atom whose acceleration matches that of rockets B and C. As a result, the string as a whole feels that it has been stretched. When a weak string is stretched, it breaks. So, this must be what happens here. In other words "yes, the string really does break."

5-8 through 5-11. As previously advertised, I won't write up solutions for these until after Spring Break. That will give you more time to think about them. They're nice things to think about while you're relaxing over the vacation ....