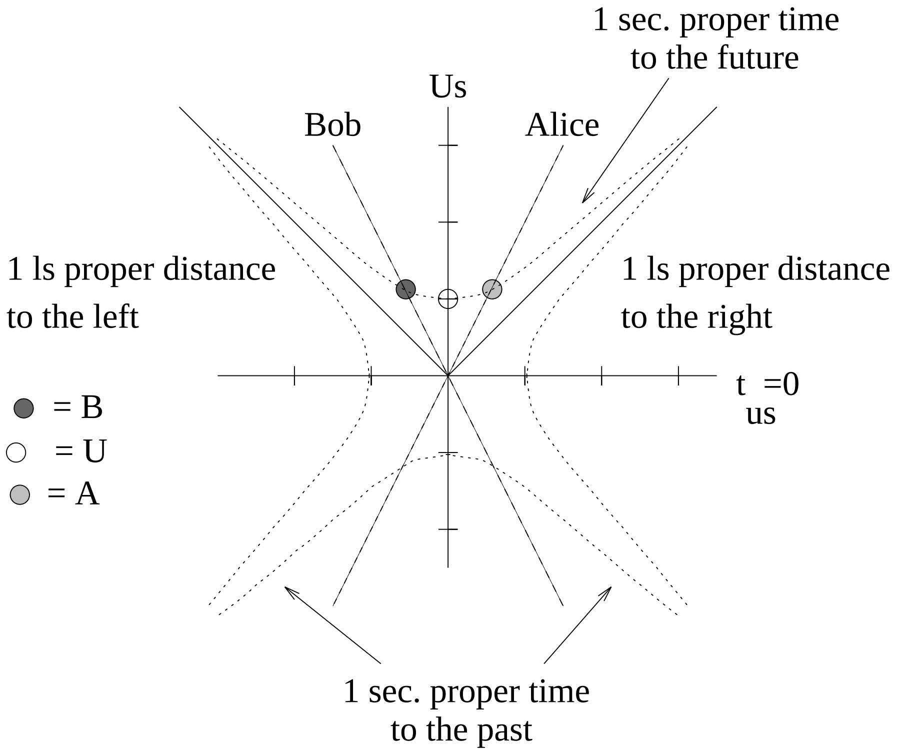
Solutions to Homework Assignment #5 – PHY312

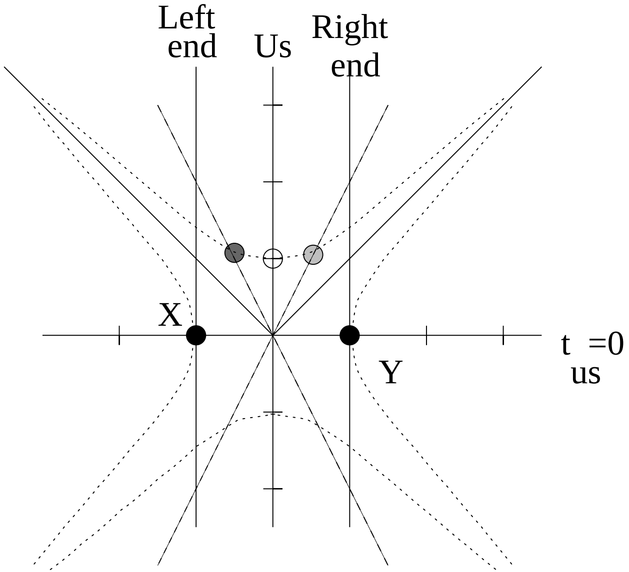
4-2. This problem is just practice drawing the kinds of things that we have been discussing in class. Much of it we have done before, but now we add in the hyperbolae showing the sets of events with given proper time and proper distance from the explosions. Remember that the set of events with given proper time τ from $(t = 0, x = 0)$ satisfies $\tau^2 = t^2 - x^2/c^2$ and is just the hyperbola with asymptotes $x = \pm ct$ which passes through the points $(x = 0, t = \pm\tau)$. Proper distance is much the same.

As a result, the diagram (for parts a, b, and c) looks like this:



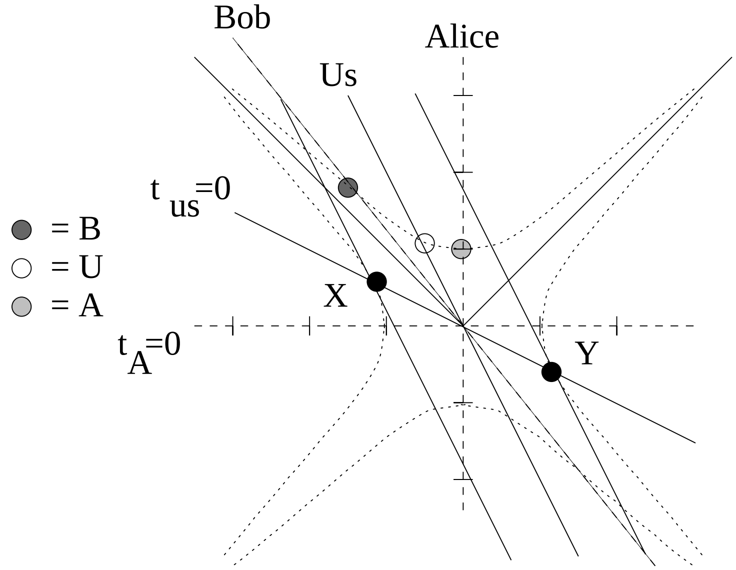
Here I have drawn our world line and $t = 0$ line using solid lines, while I have used dashed lines for the worldlines of both Alice and Bob. The outgoing rays of light are also drawn as solid lines, while the hyperbolae showing proper time and proper distance are drawn using dotted lines. The events A, U, and B are marked on the graph using the shading code indicated (A is light grey, U is white, and B is dark grey). Since our clock, Alice's clock, and Bob's clock all read zero at the origin, these clocks will all tick one second on the dotted line of events that are one second of proper time away from the origin. So, the events A, U, and B are all located at the places where the various worldlines cross the $\tau = +1$ hyperbola.

d) Since the above diagram is getting pretty crowded, I'm going to draw the diagram again, removing a few of the text labels. Since we are holding the rod, the ends of the rod are at rest relative to us. So, they will be represented by vertical lines that go straight up the diagram. Each end will be one light-second away from us, since we are in the middle. These are the two new solid lines below. Note that they intersect the left and right hyperbolae along our $t_{us} = 0$ line. This is because, along that line, the proper distance of each end from the origin is one light second. These intersection points are just the events X and Y.



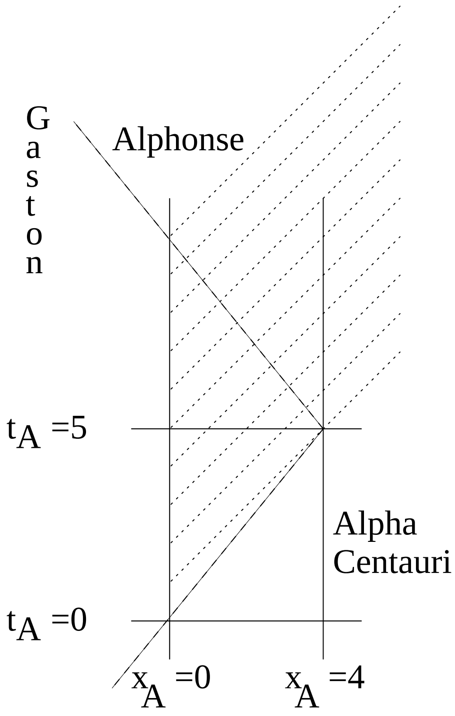
4-3. So now we just switch everything to Alice's frame of reference .... I'll use the same shading scheme as above, so that the solid lines are still our worldline and the $t_{us} = 0$ line. I've added a dashed line of simultaneity ($t_A = 0$) for Alice. The important thing here is that the hyperbolae look exactly the same from Alice's reference frame, and that the events A, U, and B are still where the upper hyperbola crosses the various worldlines.

Also, our line of simultaneity $t_{us} = 0$ is no longer flat (horizontal), but the events X and Y are still each a *proper* distance of one light-second away from the origin and so must still occur where our line of simultaneity $t_{us} = 0$ crosses the left and right hyperbolae. Once we realize this, it is easy to draw the ends of our stick. They move at the same velocity as we do (so their worldlines are parallel to ours on the diagram) and one of them passes through event X while the other passes through event Y. I have drawn these as solid lines on the diagram. Note that the worldlines of the stick ends are tangent to the hyperbolae in both diagrams. Notice also that both of these lines intersect Alice's $t_A = 0$ line *inside* the hyperbolae. This shows that Alice measures the stick to be shorter and is another way to see the length contraction effect.



4-4. For this problem, all that we have to do is to draw a bunch of light rays on the diagram describing Gaston's trip and then interpret the results.

a) I have drawn Alphonse, the star Alpha Centauri, and their lines of simultaneity using solid lines, while I have drawn Gaston's worldline as a dashed line and the light rays as dotted lines. Since Alphonse measures the star to be 4 light years away, and Gaston travels at the 80% of the speed of light, Gaston arrives at Alpha Centauri after 5 years (according to Alphonse). If Alphonse were to emit one light pulse each year, he would therefore emit 5 light pulses before $t_A = 5$ and 5 light pulses between $t_A = 5$ and Gaston's return. The diagram looks like this:

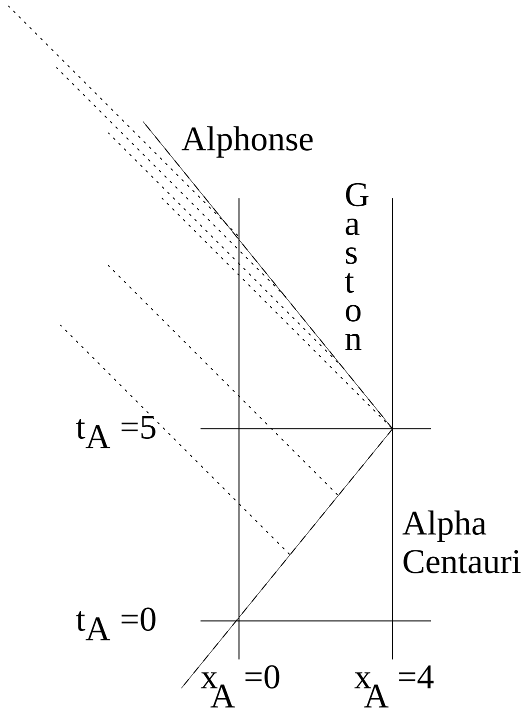


Now we have to interpret the diagram. Note that only one of the pulses actually *reaches* Gaston while he is going out, and this pulse doesn't reach reach Gaston until Gaston is just ready to turn around! So, if he is watching Alphonse through a telescope, he sees the first year of Alphonse's life stretched out over the entire time period, which is three years according to Gaston's own clock. So, he would see Alphonse moving around, doing things, and aging at one-third of the usual rate. Notice that this is due to *two* effects from Gaston's point of view: one is that he finds Alphonse to be moving away from him at nearly the speed of light, so that there is a longer and longer delay as the light takes a longer and longer time to cover the increasing distance between Alphonse and himself. However, even after Gaston corrects for this fact by subtracting out the time it took the light to travel to him (and therefore producing a set of real *measurements* of Alphonse's life), he would still find Alphonse to be aging more slowly (by a factor of $\sqrt{1 - v^2/c^2} = 3/5$ – a much less severe slowing down than 1/3). It is this remaining effect that we have called 'time dilation.'

And what happens on the way back? On the way, back, 9 light pulses intercept Gaston in a time period that, according to Gaston's clock, is only three years!!! So, Gaston would clearly *see* Alphonse to be running around very rapidly, getting up, going to bed, getting older, at *three times* the usual rate!!! However, again Gaston would naturally say that this is because Alphonse is now approaching him rapidly, so that Alphonse is chasing the light rays that he is sending out, compressing the spacing between the light rays so that they arrive too fast. Gaston might then decide to correct for this by subtracting out the light travel time again and constructing another set of *measurements* of Alphonse's life. If he does this, it turns out that he will once again find Alphonse to be aging *more slowly* than he (Gaston) is himself *just like on the way out*!!! After subtracting out the light travel time, he will again measure Alphonse to be slowed down by the time dilation rate 3/5.

The subtle point is that, once he subtracts out the light travel time, it is only easy for him to describe the first part $[(3years)(3/5) = (9/5)years]$ of Alphonse's life and the last $(9/5)years$ of Alphonse's life. It is difficult for Gaston to assign a time (t_G) to the middle $10 - 18/5 = (32/5)years$ of Alphonse's life. Actually, for this part of Alphonse's life, it is difficult for Gaston to decide how far away Alphonse 'really' was in order to figure out how much time to subtract! You may want to think for yourself about various ways (in addition to the one we discussed in class) that Gaston might try to do this – you'll find that all of them have some rather odd features. Nonetheless, Gaston still gets to watch this period of Alphonse's life, and the corresponding light rays will arrive at Gaston's telescope on the homeward journey.

b) OK, now let's look at what Alphonse sees. For this, we can use the same basic diagram but now we just show the light rays coming off of Gaston's worldline. Remember that, for Gaston, only three years pass on the way out and only three years pass on the way back, so we should only draw three light rays leaving each segment of Gaston's worldline.

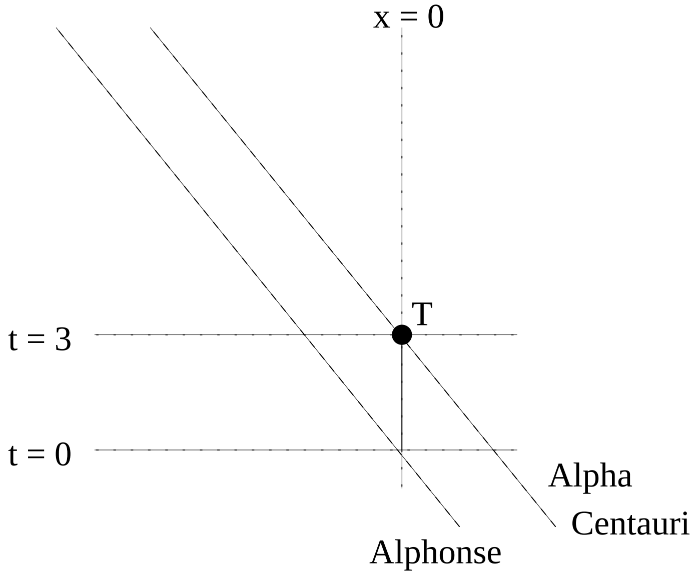


Here we see that, in the first (roughly) 9 years of Gaston's life, Alphonse intercepts only 3 of the light pulses emitted by Gaston. So, Alphonse would *see* Gaston as slowed down by a factor of 1/3 if he watched Gaston through a telescope (isn't symmetry wonderful???). Again, if Alphonse subtracted out the light travel time to make measurements of Gaston's trip, we would find Gaston to be aging slowly, but only by a factor of 3/5.

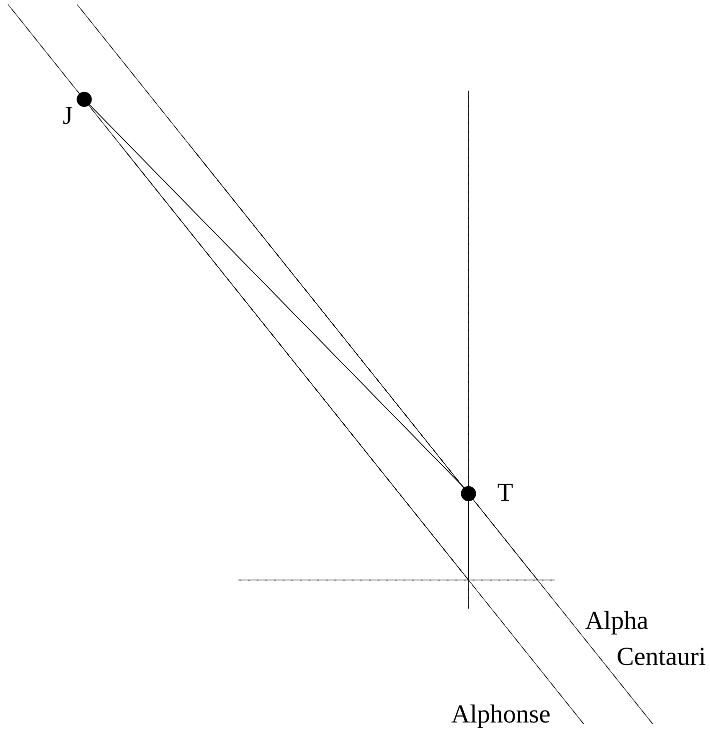
However, in the last year before Gaston returns, Alphonse intercepts three of the light rays from Gaston. So, if Alphonse was watching Gaston through a telescope, he would *see* Gaston running around very quickly, sped up by a factor of three. Nevertheless, if he again subtracts out the light travel time to make *measurements* of Gaston's life, he would find Gaston to be slowed down by the usual time dilation factor (3/5).

As an additional comment, I want to mention that these light rays point out yet another interesting way in which Alphonse's description and Gaston's are not symmetric. Note that Alphonse's light rays arrive at Gaston slowly while Gaston is going out, but quickly while Gaston is coming back. This means that Gaston notices a change as soon as he fires his rocket and turns around, *which is exactly half-way through the trip*!! On the other hand, Gaston's light rays arrive at Alphonse slowly until long after the half-way point (because the light takes some time to travel from the turnaround event back to Alphonse). It is only after 9 years that Alphonse suddenly finds Gaston's light rays to be arriving quickly.

4-5. [The optional problem.] The point of this problem is just to verify that everything works out the same in another *inertial* reference frame. Let's start by drawing a diagram. I will draw the $x = 0$ and $t = 0$ lines as dotted lines. We are to use the reference frame that moves with Gaston during his outward journey. Thus, in this frame, Alphonse and Alpha Centauri are *always* moving at $.8c$. Also, Alphonse and Alpha Centauri are $(4 \textit{light years})\sqrt{1 - v^2/c^2} = (12/5) \textit{light years}$ apart. I have drawn Alphonse and Alpha Centauri as dashed lines. The first part of Gaston's trip is drawn as a straight vertical line, since he is actually *in* this inertial frame during that period. Recall that this part of the trip takes three years according to Gaston, so Gaston will stay at $x = 0$ up until $t = 3$. Event T marks the "turnaround" and Gaston's worldline for the first part of the trip is the solid line on the diagram below:



Now, the question is, how do we draw in the second part of Gaston's trip? There are several ways to figure this out, but let's do it by finding the *time* (in this reference frame) when Gaston and Alphonse rejoin. Recall that this happens when ten years have passed for Alphonse. The time that we assign to this event must be greater than $10 yrs.$, because we must find Alphonse's clock (moving with respect to us at $.8c$) to be ticking too slowly. By the usual formula, we must assign a time of $10 yrs./\sqrt{1-v^2/c^2} = (50/3) yrs.$. That is, in our reference frame, Alphonse and Gaston do not rejoin until almost $17 yrs.$ after their parting! So, they must rejoin at event J which lies on Alphonse's worldline at $t_{us} = 16\ 2/3$. We can then draw in the worldline for the second half of Gaston's trip by connecting event J with event T. The result is:



Note that, on the way back, Gaston is moving at nearly the speed of light relative to this reference frame. In fact, since event J is at $t = 50/3$ and Alphonse is moving away at $(4/5)c$, event J must be located at $x = vt = -40/3$ *light years*. On the return trip, Gaston leaves $x = 0$ at $t = 3$, so he travels the $40/3$ *light years* to event J in $41/3$ years. Thus, relative to us, Gaston travels at $v = \Delta x/\Delta t = (40/3)(3/41)c = (40/41)c$. By the way, it is reassuring to notice that:

$$\frac{4/5+4/5}{1+(4/5)^2} = \frac{8}{5}\frac{1}{1+16/25} = \frac{8}{5}\left(\frac{41}{25}\right)^{-1} = \frac{8}{5}\frac{5^2}{41} = \frac{40}{41}.$$

That is, since Alphonse travels relative to us at $(4/5)c$, and (on the way back) Gaston travels relative to him at $(4/5)c$, our formula for the composition of velocities also tells us that Gaston's velocity relative to us is $(40/41)c$.

(For any of the math folks out there, you may have noticed that this generalizes to give an algorithm for generating and infinite number of Pythagorean triples ...)

So anyway, how much proper time passes for Alphonse and Gaston? Well, Alphonse travels along a straight path between $(t = 0, x = 0)$ and event J at $(t = 50/3, x = -40/3)$. So, the proper time is $\sqrt{(50/3)^2 - (40/3)^2} = (10/3)\sqrt{5^2 - 4^2} = (10/3)\sqrt{25 - 16} = 10$. Whew!! So, all of the messy stuff canceled out and we reach the same conclusion we had before: that a proper time of ten years passes for Alphonse during the trip!

How about for Gaston? Well, there is three years' worth of proper time in the first segment. In the second, Gaston goes from $(t = 3, x = 0)$ to $(t = 50/3, x = -40/3)$. So, for the second part the proper time is $\sqrt{(41/3)^2 - (40/3)^2} = (1/3)\sqrt{1681 - 1600} = (1/3)\sqrt{81} = 3yrs$. Again, working things properly in this reference frame yields exactly the same conclusion that we arrived at earlier. (Isn't consistency great????)

4-6. a) This problem is just a little algebra. Let's see:

$$\cosh^2\theta = \frac{1}{4}(e^{\theta} + e^{-\theta})^2 = \frac{1}{4}(e^{2\theta} + 2 + e^{-2\theta}),$$

and

$$\sinh^2\theta = \frac{1}{4}(e^{\theta} - e^{-\theta})^2 = \frac{1}{4}(e^{2\theta} - 2 + e^{-2\theta}).$$

So, if we subtract these two expressions, the $e^{2\theta}$ and $e^{-2\theta}$ terms cancel out, while the ± 2 terms add together. The result is:

$$\cosh^2\theta - \sinh^2\theta = 1.$$

b) This one is just a bit more algebra. Let's follow the hint and calculate the numerator and denominator separately.

Note that $(2\cosh\theta_1)(2\sinh\theta_2) = (e^{\theta_1} + e^{-\theta_1})(e^{\theta_2} - e^{-\theta_2}) = e^{\theta_1+\theta_2} - e^{\theta_1-\theta_2} + e^{-\theta_1+\theta_2} - e^{-(\theta_1+\theta_2)}$. Now, the second term in the numerator is just the same thing with θ_1 and θ_2 switched, so the numerator as a whole is:

$$2e^{\theta_1+\theta_2} - 2e^{-(\theta_1\theta_2)} = 4\sinh(\theta_1 + \theta_2).$$

Now we'll do the denominator.

$$(2\cosh\theta_1)(2\cosh\theta_2) = (e^{\theta_1}+e^{-\theta_1})(e^{\theta_2}+e^{-\theta_2}) = e^{\theta_1+\theta_2}+e^{\theta_1-\theta_2}+e^{-\theta_1+\theta_2}+e^{-(\theta_1\theta_2)}.$$

However

$$(2\sinh\theta_1)(2\sinh\theta_2) = (e^{\theta_1}-e^{-\theta_1})(e^{\theta_2}-e^{-\theta_2}) = e^{\theta_1+\theta_2}-e^{\theta_1-\theta_2}-e^{-\theta_1+\theta_2}+e^{-(\theta_1\theta_2)}.$$

Adding these two together, the full denominator is $2e^{\theta_1+\theta_2} + 2e^{-(\theta_1+\theta_2)} = 4\cosh(\theta_1+\theta_2)$. We therefore have

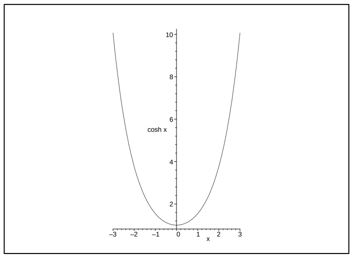
$$\frac{\tanh\theta_1 + \tanh\theta_2}{1+\tanh\theta_1\tanh\theta_2} = \frac{4\sinh(\theta_1+\theta_2)}{4\cosh(\theta_1+\theta_2)} = \tanh(\theta_1+\theta_2).$$

c) Before getting into the nitty-gritty, let's just have a look at the hyperbolic cosine function: $\cosh\theta = \frac{1}{2}(e^{\theta}+e^{-\theta})$. Notice that it is *symmetric*, meaning that $\cosh\theta = \cosh(-\theta)$. This means that the function looks just the same for negative θ as for positive θ. So, we can get away with just thinking through the positive θ part carefully and then drawing the same thing for negative θ.

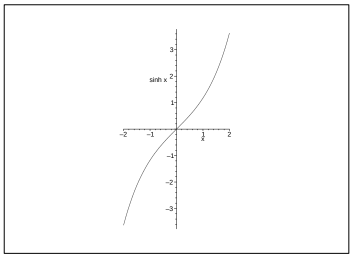
Next, let's check a simple case, say $\theta = 0$. Since $e^0 = 1$, we find $\cosh(0) = \frac{1}{2}(1+1) = 1$. This gives us one point to plot easily. Also, note that at $\theta = 0$ the two terms (e^{θ} and $e^{-\theta}$) are the same size. Now, what happens as θ gets bigger? Well, e^{θ} gets big really fast (exponentially fast!) while $e^{-\theta}$ get small (exponentially fast). So, the curve gradually begins to look more and more like $\frac{1}{2}e^{\theta}$ (which is getting really big). In other words, $\cosh\theta$ grows quite rapidly.

One might wonder if $\cosh\theta$ is always increasing for $\theta > 0$, or if it goes downward a bit before it finally starts to get bigger and bigger..... We can check this out by taking a derivative. Note that $\frac{d}{d\theta}\cosh\theta = \sinh\theta$, which is positive for $\theta > 0$! So, indeed, $\cosh\theta$ is a monotonically increasing function for $\theta > 0$.

The above discussion gives me plenty of information to draw a rough sketch of $\cosh\theta$. However, computers being what they are, I may as well just ask mine to plot a graph for me and include it in the homework solutions. You can quickly see that it has all of the properties that I just mentioned.



d) We could discuss the graph of $\sinh\theta$ along much the same lines. Note however that $\sinh(0) = \frac{1}{2}(e^0 - e^{-0}) = \frac{1}{2}(1-1) = 0$. In fact, sinh is an *antisymmetric* function, satisfying $\sinh(-\theta) = -\sinh\theta$. So, the graph for negative values of θ is just the negative of the graph for positive values of θ. Much as with cosh, for large positive θ we find $\sinh\theta = \frac{1}{2}e^\theta$, while for large negative θ (where $-\theta$ is positive), we have $\sinh\theta = -e^{-\theta}$. Once again, I'll ask the computer to draw a graph:



e) Well, $\tanh\theta = \frac{\sinh\theta}{\cosh\theta}$. Using what we found above, we can conclude that

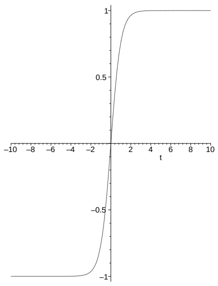
$$\tanh(-\theta) = \frac{\sinh(-\theta)}{\cosh(-\theta)} = \frac{-\sinh\theta}{\cosh\theta} = -\tanh\theta.$$

So, like sinh, hyperbolic tangent is antisymmetric. Also, we find $\tanh(0) = 0$.

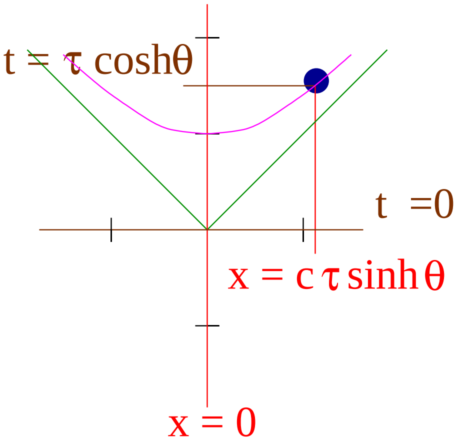
However, for large θ we find a very different behavior for tanh than we found for sinh and cosh. Recall that for large θ we have $\cosh\theta \approx \frac{1}{2}e^\theta$ and $\sinh\theta \approx \frac{1}{2}e^\theta$. Thus, for large θ we have:

$$\tanh\theta \approx \frac{\frac{1}{2}e^\theta}{\frac{1}{2}e^\theta} = 1.$$

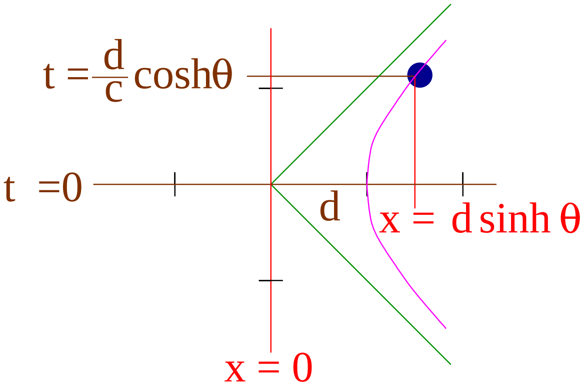
So, hyperbolic tangent does *not* grow for large positive θ. Instead, it asymptotes to 1. For large negative θ, it asymptotes to -1. It is easy to check that in fact tanh always stays between -1 and 1. Using this information, we can easily sketch the graph.... but the computer generated version looks like this:



f) Hmmm.... these equations look somewhat familiar.... Heck, let's compute $c^2t^2 - x^2 = (c\tau)^2(\cosh^2\theta - \sinh^2\theta) = (c\tau)^2$. Wow, it's a constant! So, this is just a curve of constant proper time from the origin! Let's see, is it to the past or to the future of the origin... Well, $t = \cosh\theta > 0$, so it must be to the future of the origin. So, this is just one of my constant proper time hyperbolae!



g) This is pretty much the same but we now have $x^2 - c^2t^2 = d^2(\cosh^2\theta - \sinh^2\theta) = d^2$, so this is a curve of constant proper distance from the origin. Since $x = \cosh\theta > 0$, it is to the right of the origin.



4-7. OK, recall that boost parameters (θ) are related to the relative velocity (v) by $\theta = \tanh^{-1}(v/c)$.

So, since the relative velocity between you and Alice is $c/2$, this is equivalent to a boost parameter $\theta = \tanh^{-1}(1/2) = .55$. The boost parameter between Alice and Bob is the same.

Now, boost parameters really do add together in the naive way, so the boost parameter between you and Bob is $.55 + .55 = 1.1$. Therefore, the relative velocity between you and Bob is $v = c\tanh(1.1) = .8c$; just the answer we would have gotten by combining $c/2$ and $c/2$ using the relativistic composition of velocities formula.