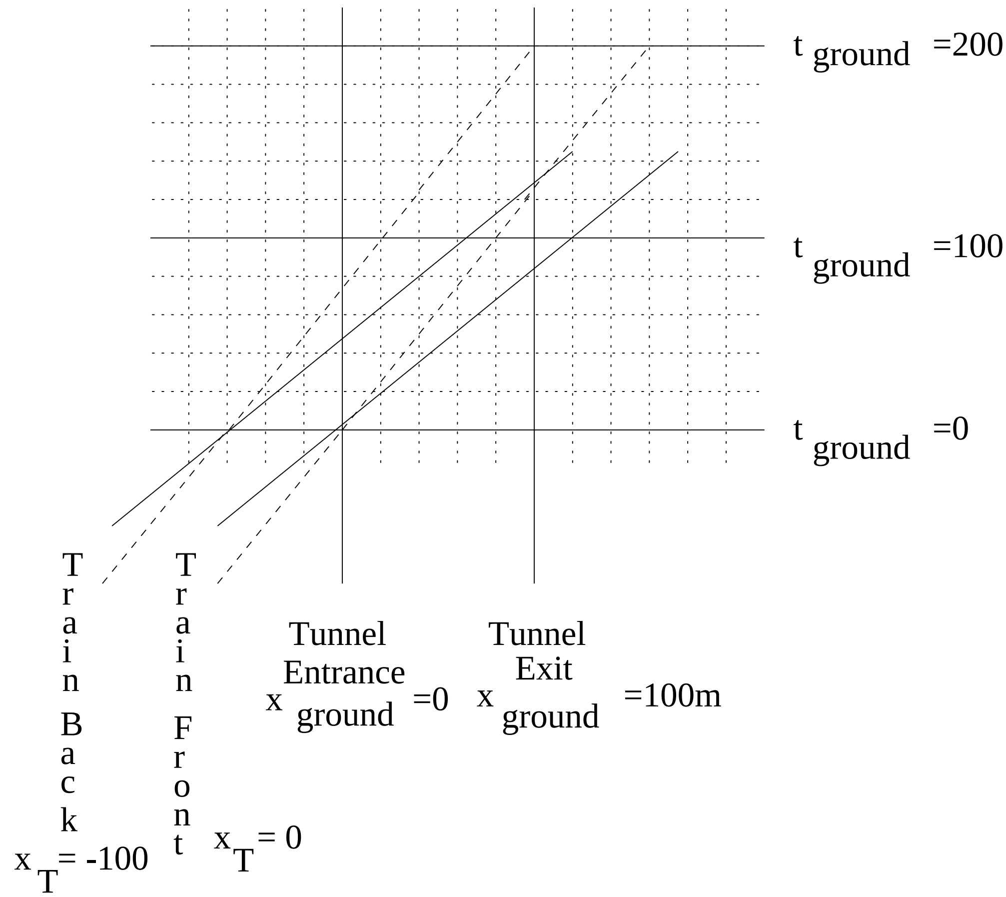
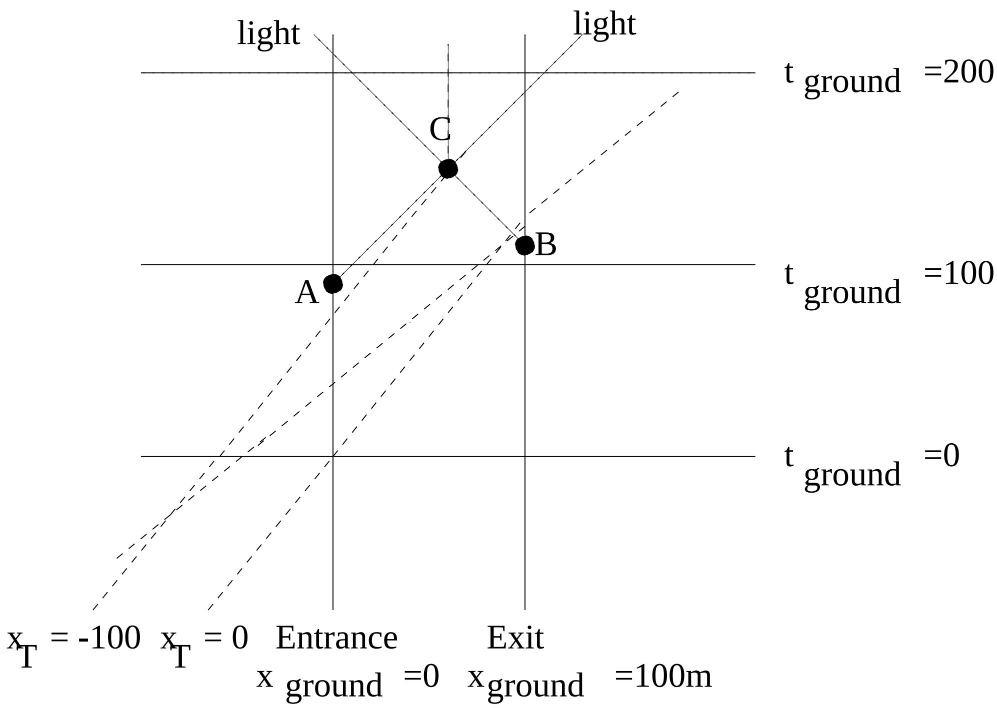
Solutions to Homework Assignment #4 – PHY312

3-9. OK, for this one, it's tricky to get the diagram just exactly right, so I've used my computer to draw lines that are exactly straight and have exactly the proper slope. I also wanted to show you the grid lines (graph paper) for this one so that you can verify that my diagram is accurate. Before we even get to the complicated stuff about the explosions, let me start by just drawing a diagram that represents the train going all the way through the tunnel with no obstructions. I'll draw this in the frame of reference of the ground (i.e., the frame of reference of the tunnel), but a similar picture could be drawn in the train's reference frame. Each (dotted) grid line below represents $20m$ of distance, or, in the time direction, the amount of time it takes light to travel $20m$. The labels are in units of meters for distance and the time it takes light to travel one meter (1 m/c) for time. Thus, the label $t = 200$ means $t = 200m/c$. As usual, I have chosen units such that the speed of light is one and light rays will be 45 degree lines on my diagram. The result looks like:



The dashed lines are the world lines of the front and back of the train, and the two solid lines sloping up and to the right are lines of simultaneity for the train. Looking along the upper one (which passes through the back of the train at $t_{ground} = 0$, it is clear that, on this line of simultaneity, the back of the train has not yet entered the tunnel while the front of the train has already gone out the exit. Thus, the train finds itself to be *longer* than the tunnel. In contrast, if we look at the line $t_{ground} = 100$, then we find that, along this line of simultaneity, both the back and the front of the train are inside the tunnel. So, the tunnel finds that the train is shorter and fits inside.

Now we can add the explosions and their effect on the train. The robber at the tunnel entrance blows that end up just after the back of the train enters. Let's call this event A. The robber at the exit blows that up just before the front of the train arrives there – let's call this event B. I have drawn these on the diagram below. The third dashed line is a line of simultaneity for the train (during the period that it is moving at $.8c$) and illustrates the fact that, from the train's point of view, event B occurs before event A. To clean up the diagram, I have removed the small grid lines. I have also drawn in a light ray leaving each of these events, representing the light that comes out from the explosions and showing where the *information* that the explosions have occurred actually reaches the front and back of the train. The light rays are the lines at 45 degrees from the vertical.



Note that, if the explosions are at events A and B exactly as I have drawn them, the back of the train finds out about both explosions at exactly the same time!!!! (At event C.) However, this is just due to exactly where I chose to put events A and B. Event B could just as well have been a bit earlier or a bit later in which case the signals would reach the back of the train at different times.

Now, the front of the train hits the rubble at the exit just above event B and, presumably, stops. However, as we just said, the back of the train has no knowledge of the explosions until event C, and so cannot possibly stop before then. In particular, the pressure wave running through the train that constitutes the cars at the front of the train pushing on the cars behind them, trying to get them to stop, will not reach the back of the train until event C. Soon after event C, of course, the train will stop, presumably with a great shattering of class and twisting of metal, since the 100m train (which is now at rest), is now occupying a region of space that is only about 40m long. (Yes, $40m$ long – you can read this off of the diagram.)

Because the back of the train will stop shortly after event C, I have drawn the worldline of the back of the train with a corner in it, so that it becomes straight vertical at event C. This shows that, after event C, the back of the train is at rest, but is only $40m$ away from the front of the train!!!!!!!

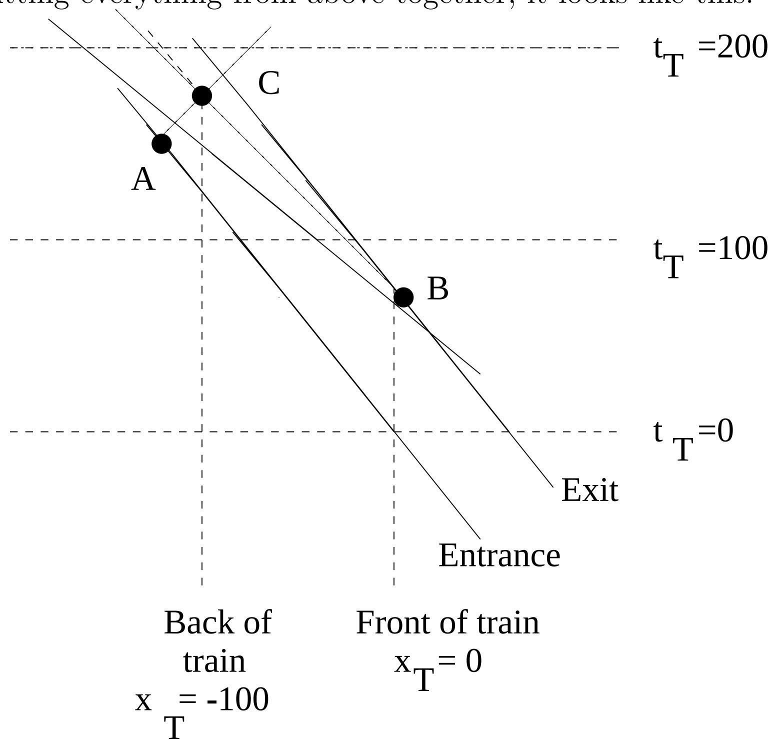
To say this all in words:

From the point of view of the robbers, the first thing that happened was that the front of the train entered the tunnel, then the back of the train entered. Then, just after that, one of the robbers blew up the tunnel entrance at event A. A short time after that, the other robber blew up the exit at event B. However, since the back of the train couldn't find out that either end had been blown up until event C, it kept on moving. Only when it was $40m$ away from the exit did it finally get hit by the car in front of it and stop. The train has been stopped and the safe has probably been blown open, too! Of course, the gold may have all been vaporized .....

Now, what about from the train's point of view? We can also read this off of our diagram, since we drew in a line of simultaneity for the train. We can see that, from the train's perspective (at the front), the first thing that happens is that the front of the train enters the tunnel. Then, *before* the back of the train enters, the exit of the tunnel is blown up (at event B), and the front of the train slams into the rubble. At this point, the front of the train is forced into the same reference frame as the rubble, ground, and tunnel. However, the back of the train doesn't know about this yet, so it keeps on moving and enters the tunnel. (Meanwhile, of course, the cars in between the front and back are running into each other, smashing apart, and generally "finding out about event B." Finally, the back of the train finds out about the explosions at event C and has its reference frame forcibly altered as well.

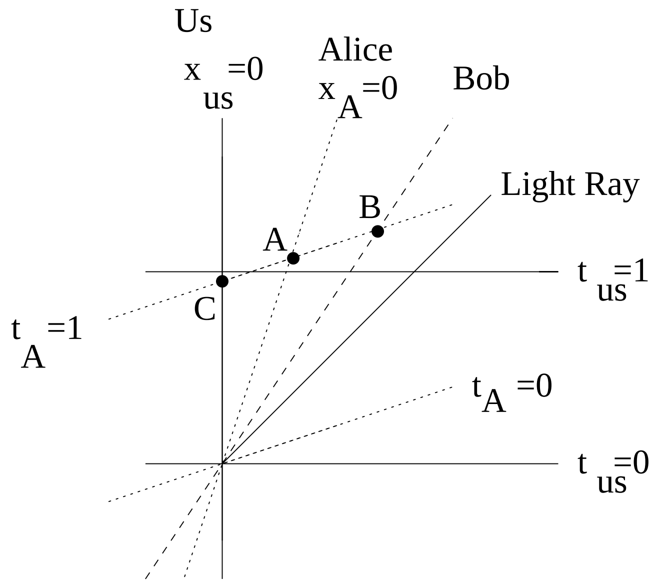
From the perspective of the back of the train, the story is much the same, but I want to emphasize that a passenger at the back of the train would just be merrily riding along, entering the tunnel, when suddenly, at event C, the passenger *sees* the light from *both* explosions. Just after this, the car that is second from the back slams into the back car as the shock wave arrives from the collision of the front of the train with the rubble at the exit. The reference frame of our passenger is then altered radically by the huge force of that collision.

Our last task is to draw a diagram from the inertial reference frame of the (moving) train – that is, the reference frame of the train before it is stopped. Putting everything from above together, it looks like this:



Again, the solid lines represent the tunnel entrance and exit, and a line of simultaneity for the tunnel. The dashed lines represent the worldlines of the front and back of the train, and lines of simultaneity in the reference frame that the train had while it was moving at $.8c$. I used the computer and grid again to draw this one, so it should be accurate. Note that, in this frame, the exit is blown up at event B which is about $t_{train} = 75$, while the entrance is blown up at event A which is at $t_{train} = 150$. After event A, the front of the train 'stops,' that is, it has zero velocity relative to the exit, and it follows the worldline of the exit. Similarly, at event C, the back of the train changes reference frames. For this reason, I have drawn the worldline of the back of the train to bend to the left at event C.

3-10. So, we have three inertial frames: Alice, Bob, and ourselves. We have said that v_A is the speed at which Alice moves away from us, v_{BA} is the speed of Bob as measured by Alice, and v_B is Bob's speed away from us (as measured by ourselves). We have the following diagram:



indicating Alice's line of simultaneity $t_A = 1$, and the event B where this line intersects Bob's worldline. I have also labelled the event C where the line $t_A = 1$ crosses our worldline. This will be useful below.

a) We want to find the coordinates of event A in our reference frame. Let us choose units so that $c = 1$ for convenience. Recall that event A is the event of Alice's clock ticking $1sec$. Now, our discussion of time dilation tells us that the time (t_{us}) that we assign to this event is *bigger* than one second by a factor of $\frac{1}{\sqrt{1-v_A^2}}$ (since $c = 1$). Thus, $t_{us} = \frac{1}{\sqrt{1-v_A^2}}$.

How about x_{us}? Well, event A is on Alice's worldline and, at event A, Alice has been moving away from us at velocity v_A for a time $t_{us} = \frac{1}{\sqrt{1-v_A^2}}$. Thus, Alice and event A are at position $x_{us} = v_A t_{us} = \frac{v_A}{\sqrt{1-v_A^2}}$.

b) We want to write down the equation of this line ($t_A = 1$) in terms of our coordinates. We can do this by using the fact that we can find the slope of the line, and we can find the t_{us} value of event C where this line crosses our t-axis (the line $x_{us} = 0$). Let's find the t_{us} value of event C first.

Recall that the line $t_A = 1$ is the set of events that Alice finds to occur at time 1. Event C is the event where *our* clock is 'when' (i.e., on the line of simultaneity such that) Alice says $t_A = 1$. So, the question 'What is t_{us} at event C?' is just the question 'What time does Alice read off of our clock if she records it at $t_A = 1$?' Recall that our clock is moving relative to Alice, so she finds that it runs more slowly by a factor of $\sqrt{1-v_A^2}$. So, the coordinate t_{us} at event C must be just $t_A\sqrt{1-v_A^2} = \sqrt{1-v_A^2}$. (Recall that $c = 1$ in our units.)

One way to find the slope would be to use the fact that we now know the coordinates of two events on this line. We found the coordinates of event A in part (a), and we just found the coordinates of event C. Knowing the coordinates of the two events, we can calculate the slope using

$$m = \frac{\Delta t}{\Delta x}.$$

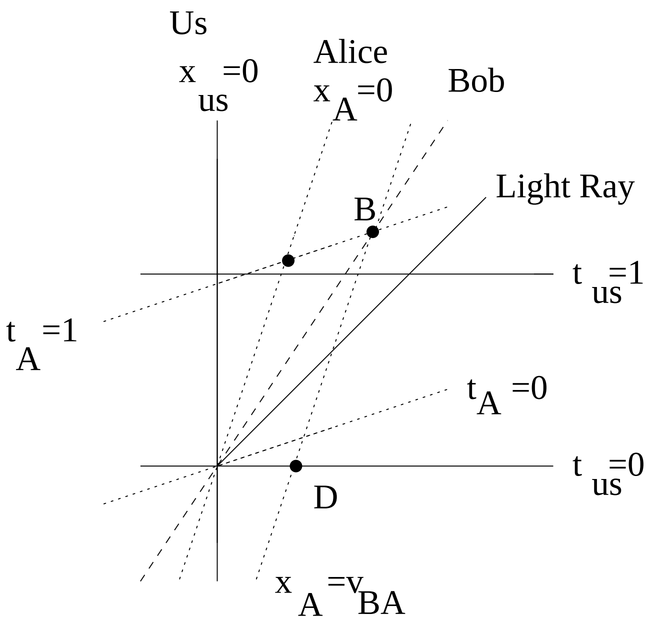
The result is $m = v_A$.

There is also another way to find the slope. We might note that the line of interest is the the same angle below the light cone (which has slope 1) that Alice's worldline is above it. Thus, the line of simultaneity is just Alice's worldline "flipped over the light cone." So, the slope of the line of simultaneity is 1 divided by the slope of Alice's worldline. In other words, since Alice's worldline satisfies $\Delta x_{us}/\Delta t_{us} = v_A$, Alice's line of simultaneity must satisfy $m = \Delta t_{us}/\Delta x_{us} = v_A$.

In any case, this means that the equation of this line is $(t_{us} - \sqrt{1-v_A^2}) = v_A x_{us}$, or

$$t_{us} = v_A x_{us} + \sqrt{1-v_A^2}. \quad (1)$$

Now, we can go on to the other line that I drew through event B Recall that, in Alice's reference system, event B has coordinates $(t_A = 1, x_A = v_{BA})$. In particular, the line $x_A = v_{BA}$ goes through event B, and we want to find the equation of this line.



c) Again, we can write down the equation by knowing the slope and the coordinates of a point on the line. This time, the line is at a constant distance from Alice, and so represents an object (like the end of a stick) in Alice's reference frame. In other words, that object moves at speed v_A is our reference frame. Thus, the line will satisfy $\Delta x_{us}/\Delta t_{us} = v_A$.

To determine the rest of the equation, we can use the event D where this line intercepts the x_{us}-axis (our line $t_{us} = 0$). Note that we can think of the line $x_A = v_{BA}$ as representing one end of a rod of length v_{BA} that Alice holds in front of her (such that she is at the back end of the rod). At $t_{us} = 0$, one end of the rod (where Alice is) is at $x_{us} = 0$ while the other end of the rod is at event D. Thus, the x_{us} coordinate of event D is exactly equal to the length of this rod as *we* measure it! Since this rod is moving relative to us, we find it to be shorter than Alice does by a factor of $\sqrt{1 - v_A^2}$, so we find it to have length $v_{BA}\sqrt{1 - v_A^2}$. In other words, event D has coordinates $(t_{us} = 0, x_{us} = v_{BA}\sqrt{1 - v_A^2})$. So, the line $x_A = V_{BA}$ satisfies

$$(x_{us} - v_{BA}\sqrt{1 - v_A^2}) = v_A t_{us}. \quad (2)$$

d) Well!!! Now all that is left is to do a bit of algebra! Note that equation (1) has already been solved for t_{us}. Let us then substitute the expression on the right hand side of equation (1) for t_{us} in equation (2). This gives

$$x_{us} - v_{BA}\sqrt{1-v_A^2} = v_A(v_A x_{us} + \sqrt{1-v_A^2}). \quad (3)$$

Moving the terms with an x_{us} to the left hand side and the other terms to the right yields

$$x_{us}(1-v_A^2) = (v_{BA}+v_A)\sqrt{1-v_A^2}. \quad (4)$$

Dividing by $1-v_A^2$ gives

$$x_{us} = \frac{(v_{BA}+v_A)}{\sqrt{1-v_A^2}}. \quad (5)$$

Then, we can just plug this back into, say, (1) to find that

$$t_{us} = v_A x_{us} + \sqrt{1-v_A^2} = \frac{v_A(v_{BA}+v_A)}{\sqrt{1-v_A^2}} + \sqrt{1-v_A^2}$$

$$= \frac{v_A v_{BA} + v_A^2 + (1-v_A^2)}{\sqrt{1-v_A^2}} = \frac{v_A v_{BA}+1}{\sqrt{1-v_A^2}}. \quad (6)$$

e) Finally, we can divide the above two expressions to find Bob's velocity:

$$v_{Bob} = \frac{x_{us}}{t_{us}} = \frac{v_{BA}+v_A}{1+v_A v_{BA}}. \quad (7)$$

QED

Oh, I can now put the c's back in by realizing that "v in units where $c=1$" is *exactly* v/c. So, I can just replace every v in (7) with v/c to get

$$\frac{v_{Bob}}{c} = \frac{v_{BA}/c + v_A/c}{1+v_A v_{BA}/c^2} \quad (8)$$

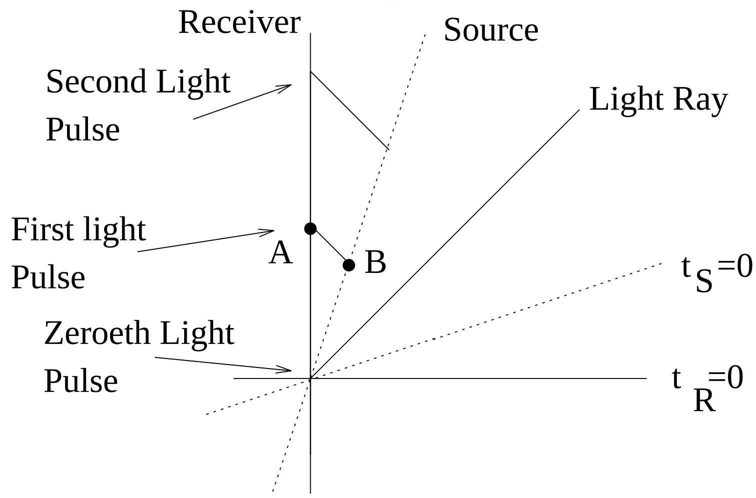
or

$$v_{Bob} = \frac{v_{BA}+v_A}{1+v_A v_{BA}/c^2}. \quad (9)$$

3-11. Well, let me dig out the old calculator here .... for $v_A = .01c$ and $v_{BA} = .01c$, I get $v_B = .019998c$, which is darned close to $.02c$. In other words, when the velocities are much less than the speed of light, velocities *almost* add in the naive way. It's only for speeds close to c that things become noticeably different.

3-12. Let's see, if $v_{BA} = c$, then I get $v_B = (v_A + c)/(1 + v_A/c) = c$. So, this equation does respect the idea that the speed of light is the same in all inertial frames!!

3-15. [Optional] For this problem, we want to understand the time τ_R (as measured by the receiver) that passes between when the light pulses are received in terms of the time τ_S (as measured by the source) between when they are sent. A spacetime diagram for this problem looks like this:



Note that the 'zeroth' light pulse is sent at $t_S = 0$ and arrives at $t_R = 0$. So, to compute the time between light pulses, all we need to know is when the first pulse is received. Note that this happens at event A.

Now, the first light pulse is sent out from event B. So, it would help to know the coordinates of this event. Event B is just the event where a clock carried by the source would 'tick' $t_S = \tau_S$. So, we can figure out the time of event B in the reference frame of the receiver by thinking back to our light clock example .... If we are in the receiver's reference frame, we would have to see the source's clock as running more slowly than the ours, so our clock will read *more* time to have passed than the sources clock by the famous factor of $1/\sqrt{1 - v^2/c^2}$. So, event B is at $t_R = \tau_S/\sqrt{1 - v^2/c^2}$.

We would also like to know the x_R coordinate of event B. But, event B lies on the worldline of the source which is traveling at speed v and which passed through the origin. So, $x_R = vt_R = v\tau_S/\sqrt{1 - v^2/c^2}$.

Now, the line connecting events B and A is traveling at the speed of light in the negative direction, so it satisfies $\frac{\Delta x_R}{\Delta t_R} = -c$. Since it passes through event B at $(t_R = \tau_S/\sqrt{1-v^2/c^2}, x_R = v\tau_S/\sqrt{1-v^2/c^2})$, the equation of this line is

$$(x_R - \frac{v\tau_S}{\sqrt{1-v^2/c^2}}) = -c(t_R - \frac{\tau_S}{\sqrt{1-v^2/c^2}}).$$

Dividing both sides by $-c$, at $x_R = 0$ this becomes:

$$\frac{v\tau_S}{c\sqrt{1-v^2/c^2}} = t_R - \frac{\tau_S}{\sqrt{1-v^2/c^2}},$$

or

$$\tau_S \frac{v/c+1}{\sqrt{1-v^2/c^2}} = t_R.$$

To make this look nicer, we can use the fact that $1-v^2/c^2 = (1+v/c)(1-v/c)$ to write this as

$$t_R = \tau_S \frac{v/c+1}{\sqrt{1+v/c}\sqrt{1-v/c}} = \tau_S \sqrt{\frac{1+v/c}{1-v/c}}.$$

And, finally, since the time interval τ_R measured by the receiver between the origin and event A is just the coordinate t_R of event A, we have

$$\tau_R = t_R = \tau_S \sqrt{\frac{1+v/c}{1-v/c}}.$$

QED

3-14. [Optional] Piece of cake: we can just use the expansions from problem 2-2. The only trick is to remember to throw away any term with a v^2 in it.

$$\mathit{Relativistic:} \quad \tau_R = \sqrt{\frac{c+v}{c-v}} = \frac{1+v/c}{\sqrt{1-v^2/c^2}} \approx (1+v/c)(1) = 1+v/c.$$

$$\mathit{Newtonian:} \quad \tau_R = \frac{1+v_S/c}{1+v_R/c} \approx (1+v_S/c)(1-v_R/c) \approx 1+(v_S-v_R)/c = 1+v/c.$$

4-1. I know it's hard to believe, but this problem is just a straightforward computation. (Gasp!!!!) All we have to do for each path is to use the formula $\Delta\tau = \sqrt{(\Delta t)^2 - (\Delta x/c)^2}$ for each straight line segment and then add up all of the proper times for each path. Oh, and the graph is drawn in units where $c = 1$ so we can just ignore the c's if we like.

A) Path A consists of two straight segments: one from (t=-4, x=0) to (t=0, x=-2) and one from (t=0,x=-2) to (t=4,x=0). For the first segment the proper time is $\sqrt{16-4} = 2\sqrt{3}$. By symmetry, it's the same for the second. So, for the entire path, the proper time is $4\sqrt{3}$ seconds.

B) Path B is a single straight segment from $(t = -4, x = 0)$ to $(t = +4, x = 0)$ going straight up the t axis. This one is especially easy: $\Delta\tau = 8$ seconds since $\Delta t = 8$ seconds and $\Delta x = 0$.

C) Path C again has two straight pieces. One goes from $(t = -4, x = 0)$ to $(t = -1, x = 1)$ and one from $(t = -1, x = 1)$ to $(t = +4, x = 0)$. For the first one, the proper time is $\Delta\tau_1 = \sqrt{9-1} = 2\sqrt{2}$. For the second, we have $\Delta\tau_2 = \sqrt{25-1} = 2\sqrt{6}$. So, the total proper time is $\Delta\tau_1 + \Delta\tau_2 = 2\sqrt{2}(1+\sqrt{3})$.

D) Finally, path D consists of 4 straight line segments. However, a look at the graph shows that all four of these paths are moving at the speed of light! Light rays have $\Delta x = \pm c\Delta t$, and therefore *zero* proper time! So, the proper time for each segment is zero, and the total proper time is zero as well!

Clearly path (D) has the least proper time (zero) while path B has the greatest ($\Delta\tau = 8$).