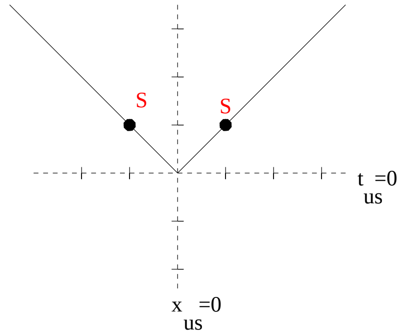
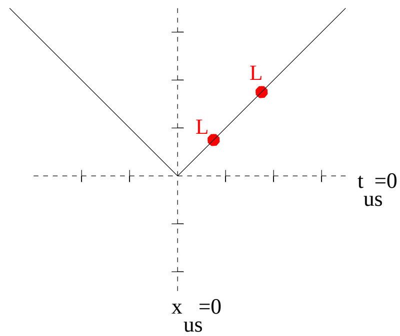
Solutions to homework Assignment #3
PHY312 – Spring, 2003

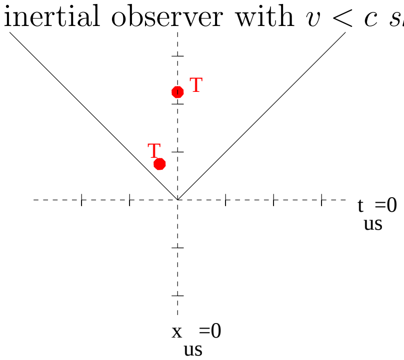
3-3. a. A pair of spacelike separated events is shown below. There are, of course, many other choices. We just need to choose a pair of events such that an inertial observer traveling at $v < c$ could *not* pass from one to the other.



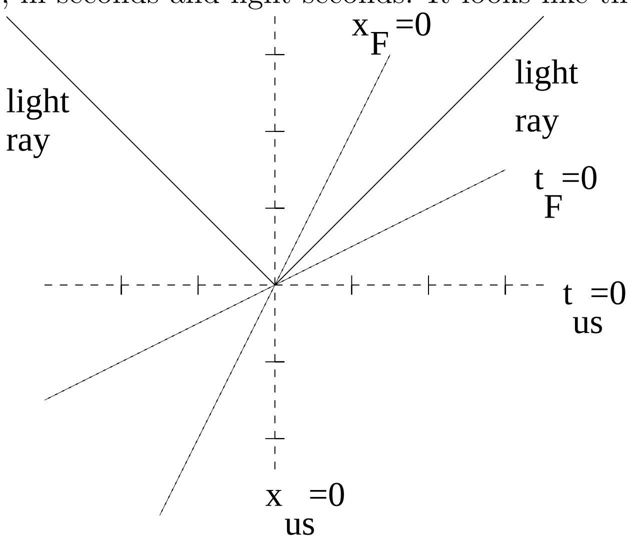
b. Here are two lightlike separated events. Again, there are many choices but the two events *must* lie on a common light ray.



c. Here are two timelike separated events. The important thing is that an inertial observer with $v < c$ *should* be able to travel between the events.

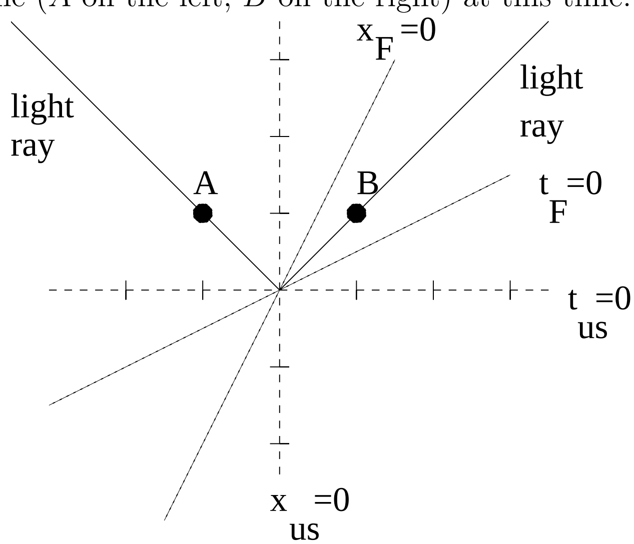


3-4. a. Well, part (a) is pretty easy. The only trick is to get the speed right. I'll put some tick marks along the t and x axes to show the scale – say, in seconds and light-seconds. It looks like this:

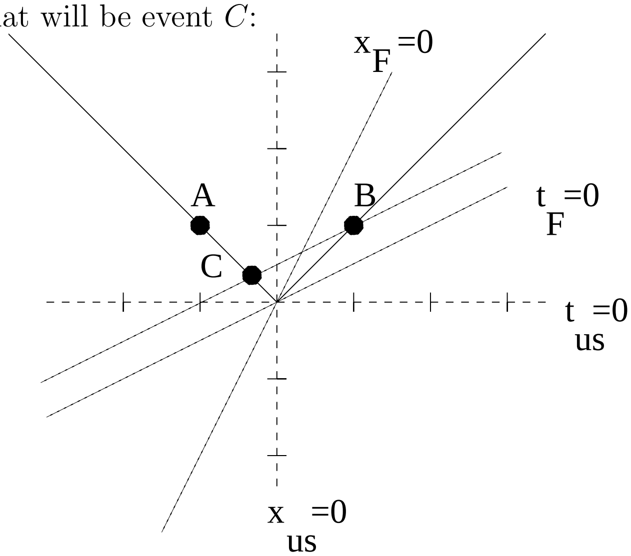


Note that, to indicate the velocity $\frac{1}{2}c$, I have drawn the $x_F = 0$ line sloping upward and to the right so that it moves over one unit of space for every two units of time that it moves up ($v = \frac{1Ls}{2sec}$).

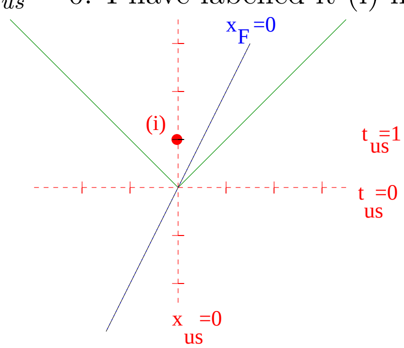
b. No problem here – just look on the above graph, find $t_{us} = 1$ (the first tick mark going up the $x_{us} = 0$ line), and mark the two events on the light cone (A on the left, B on the right) at this time:



c. Let's see, what is event C? It's the event which i) lies on the left light ray and ii) is such that our friend finds C and B to be simultaneous. Well, the set of all events that our friends finds to be simultaneous with B is given by our friend's line of simultaneity passing through event B – a line parallel to $t_F = 0$, but moved so that it passes through B. So, all we have to do is to draw this line on our diagram and find where it intersects the left light ray!! That will be event C:



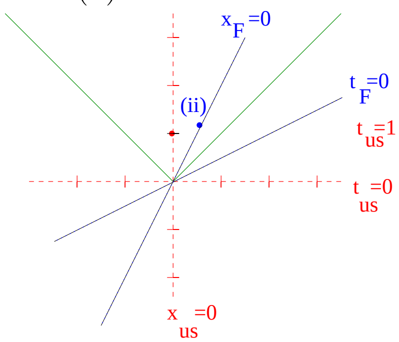
3-5. The setup here is much as in problem 3-4.
a. Let's say that my frame of reference is the same as yours, so that it is 'our' frame of reference.
(i) The event where your watch ticks $t = 1$ is easy to find: it lies at $x_{us} = 0, t_{us} = 0$. I have labelled it (i) in the diagram below.



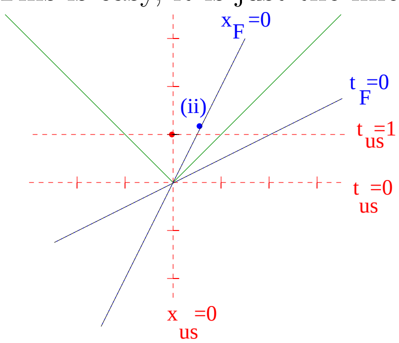
(ii) How about the event where our friend's watch ticks 1 second? (Let's call this event (ii).) It must lie on our friend's worldline. Recall that, if our friend carries a light clock, this clock moves relative to us and we see the light following a diagonal line. This means that we see the light traverse a longer path than our friend does, so that we must assign a greater time to this process: $t_{us} > t_F = 1sec$. In particular, we showed that

$$t_{us} = \frac{1sec}{\sqrt{1 - v^2/c^2}} = \frac{2}{\sqrt{3}} \approx 1.15sec$$

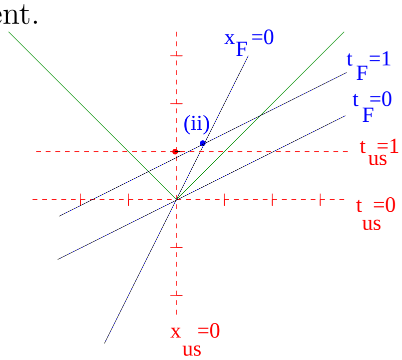
at event (ii). This information allows us to draw event F on our diagram:



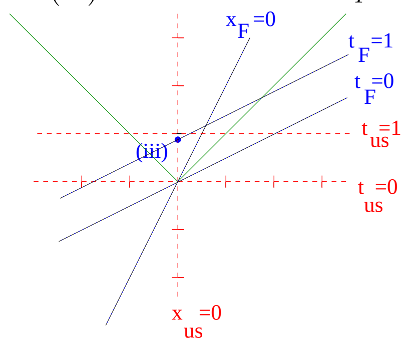
b. This is easy, it is just the line $t_{us} = 1$.



c. This is also straightforward. Since event (ii) marks the first tick of our friend's clock, we need only draw a line of appropriate slope through this event.



d. It is probably worthwhile to first mark this event on our diagram. Let's call it (iii). It is on the line $t_F = 1sec$ *and* on our worldline as shown below.



Since the event is on the $t_F = 1sec$ line, our friend says that it happened at a time of one second. There are a couple of ways to find out what time *we* assign, but the simplest is to note that this event is *on our worldline* and so can be thought of as *the event* where our clock ticks $t_{us} = T$ for some T. Now, we know that, if we carry a light clock, it moves relative to our friend so that our friend sees the light in our clock travel along a diagonal path. Thus, our friend finds the light to travel a longer distance and must assign the larger time $\frac{T}{\sqrt{1-v^2/c^2}}$. But, we already know that our friend assigns the time $t_F = 1second$! Thus, we have

$$\frac{T}{\sqrt{1-v^2/c^2}} = 1second$$

or,

$$T = 1second\sqrt{1-v^2/c^2} = \frac{\sqrt{3}}{2} \approx .86.$$

If this seems confusing, it might help to redraw the above diagram in our friend's frame of reference.

e) This part is just the mirror image of part (d). By construction, the event lies at $t_{us} = 1$. An argument just like the one in part (d) then gives $t_F = \sqrt{1-v^2/c^2}$.

3-6. The comment about the expansions should have referred to problem (2-2), not problem (3-2). The point is that the time dilation factor is

$$\frac{1}{\sqrt{1-v^2/c^2}} \approx 1 + \frac{1}{2}v^2/c^2$$

$$
\begin{aligned}
&= 1 + \frac{1}{2}\left(\frac{6\times 10^2}{3\times 10^8}\right)^2 \\
&= 1 + 2\times 10^{-12}. \qquad (1)
\end{aligned}
$$

That is, the effect is just two parts in a trillion.

3-7 a. Let's think about an (inertial) clock that moves with the muon and just happens to tick $t = 0$ when the muon is created in the upper atmosphere. The muon should decay at the event where that clock ticks $t = 10^{-6}$ seconds. The question we want to ask is what time this clock reads when it reaches the ground. If it is less than 10^{-6} seconds, the muon will still be around.

Let's see, in our frame of reference, the clock has to travel $30km$ at $c/2$. So, this will take a time

$$t = 30km/(c/2) = 2(30,000m)/(3\times 10^8 m/s) = 2\times 10^{-4} \text{ seconds.}$$

How much time passes in the clock's reference frame? Recall that, if this clock is a light clock, then when our own light clock ticks, we find that the light in the moving (muon) clock is trying to cover a longer distance and so has not yet reached the end – that is, the moving clock has not yet 'ticked.' So, the time measured by the moving clock is less by the famous factor of $\sqrt{1 - v^2/c^2} \approx .86$, and the moving clock has only measured 17×10^{-5} seconds to pass. However, this is still more than 10^{-6} and the muon must have already decayed.

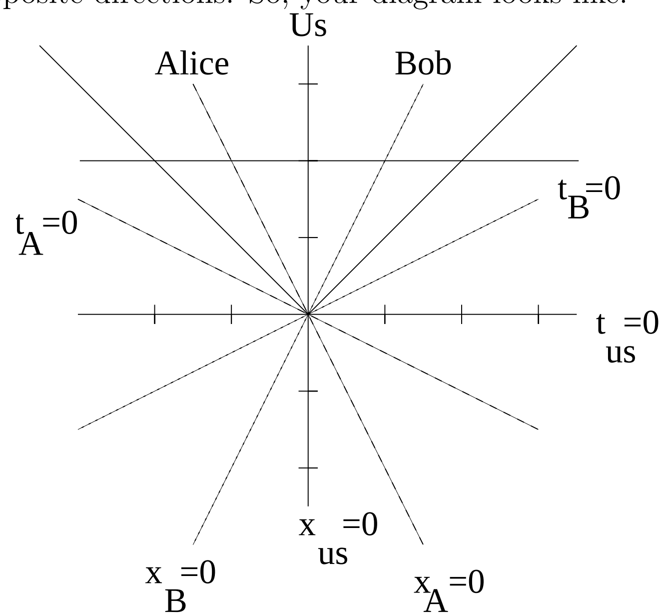
b) Now we're going to let the muon travel much faster. At $.999999c$, we find that it takes the muon 10^{-4} seconds to come down. However, now $\sqrt{1 - v^2/c^2}$ is approximately 1.4×10^{-3}. So, on the way down, only $(10^{-4}$ seconds$)(1.4\times 10^{-3}) = 1.4\times 10^{-7}$ seconds passes for the muon. This time then, the muon *is* still around when it reaches the ground.

On the way back up, another 1.4×10^{-7} seconds will pass for the muon. So, it will spend a total of 2.8×10^{-7} seconds in flight. If we want it to decay at the top of the atmosphere, we should hold it for the extra time:

$$10^{-6} \text{ seconds } - \ 2.8\times 10^{-7} \text{ seconds } = 7.2 \ \times \ 10^{-7} \text{ seconds,}$$

as experienced by the muon. But, since we are just holding the muon, we and the muon are in the same inertial frame and we all measure time in the same way. Thus, this result is also how long we hold the muon according to our own clock.

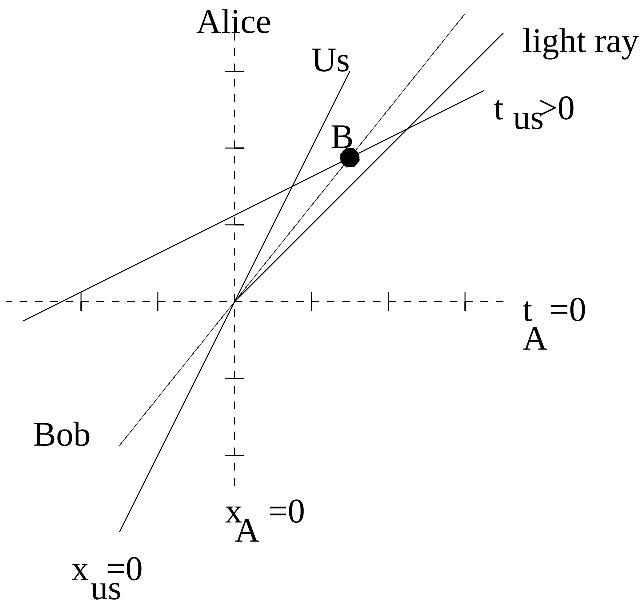
3-8. In your frame of reference, this one is a piece of cake. It's just like problem 3-4, but now we have two moving observers, each going at $c/2$ in opposite directions. So, your diagram looks like:



This time, I used solid lines both for us and for the light cone, so that I could use dashed lines for Alice and dotted lines for Bob. I have drawn in everyone's $t = 0$ lines (assuming that you agree to all set your clocks to zero at the event where you meet). I have also drawn in another of our lines of simultaneity so that we can verify that, *along that line*, Bob and Alice are halfway between us and the light cone.

Now, we want to draw the same thing in Alice's frame of reference. It's not hard to draw our worldline on her diagram, since we know that she will find us to be receding at $c/2$. But where do we put Bob's worldline?

The key point here is the observation made in the statement of the problem. At any given time (as defined by *us*), we find Bob to be halfway between you and the (right) light ray. So, we should start by drawing us on Alice's diagram, and then drawing one of our lines of simultaneity (for some time $t_{us} > 0$). Then, let's find the event (call it B) that lies on this line, halfway between us and the (right) light ray (as measured along the line). Bob's worldline must go through this event (and through the origin), so we can now draw it one the diagram:



If we like, we can now read the speed v_{BA} of Bob relative to Alice directly off of the diagram. The event B is at $x_A = 1.5$ and at $t_A = 1.875$ (my computer has a fine grid which really does let me read the numbers off to this accuracy). So, $v_{BA} = x_A/t_A = .8$ light-years per year, that is, $.8c$.

Just to summarize, if we move relative to Alice at $\frac{1}{2}c$, and Bob moves relative to us at $\frac{1}{2}c$, then Bob moves relative to Alice at $.8c$. This is exactly the answer given by a formula you might have seen in Einstein's book.

If you decided to draw in another observer moving at $\frac{1}{2}c$ relative to Bob, it is clear that this observer would still move relative to Alice at a speed less than c, although it would be quite close to c. This new worldline would be 'crunched up against the light cone' on Alice's diagram.